Differential Algebraic Equations

(LINEAR)

$$Ay' + By = c(t) \qquad y = y(t)$$

If A is non singular then $\Rightarrow$ is an ODE

$\Rightarrow$ Left multiply by A^{-1}

$$y' + A^{-1}By = A^{-1}c(t)$$

If A is singular then $\Rightarrow$ is a DAE

$\Rightarrow$ Use Kronecker normal form

DEF: (Regular pencil)

Given $A, B \in \mathbb{R}^{n \times n}$ if

$$A + \lambda B$$

is NOT singular for some values of λ

then (A, B) is a REGULAR PENCIL otherwise

is a SINGULAR PENCIL

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|A + \lambda B| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \Rightarrow$$ (A, B) is a singular pencil

Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|A + \lambda B| = \begin{vmatrix} 1 & 0 & 1 \\ \lambda & \lambda & 1+\lambda \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \lambda & 1+\lambda \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} \lambda & \lambda \\ 0 & 1 \end{vmatrix}$$

$$= \lambda - (1+\lambda) + \lambda$$

$$= \lambda - 1$$

is a polinomial in λ $\Rightarrow$ NOT IDENTICALLY 0

$\Rightarrow$ (A, B) is a REGULAR PENCIL

KRONEKER NORMAL FORM

LET (A, B) REGULAL PENCIL THEN THERE EXIST TWO MATRIX P, Q such that

$$PAQ = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix} \qquad PBQ = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix}$$

- P, Q are nonsingular
- J is in JORDAN NORMAL FORM (NOT NECESSARY)
- N is Nilpotent

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_i \end{pmatrix} \qquad J_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

$N^p = 0 \quad N^{p-1} \neq 0$ is nilpotent of order p

Example $N = \begin{pmatrix} 0 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ $\quad N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad N^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$p = 3$

LINEAR DAE IN NORMAL FOR

$$Ay' + By = c \Rightarrow P(Ay' + By) = Pc$$

$$PAy' + PBy = Pc$$ defining $Qz = y$

$$\underbrace{PA}_{\tilde{A}}Qz' + \underbrace{PB}_{\tilde{B}}Qz = \underbrace{Pc}_{\tilde{c}}$$

$$\tilde{A}z' + \tilde{B}z = \tilde{c}$$ where

$$\tilde{A} = \begin{pmatrix} I & \\ & N \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} J & \\ & I \end{pmatrix}$$

SOLVE DAE IN NORMAL FORM:

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad \begin{cases} z_1' + J z_1 = c_1 \\ N z_2' + z_2 = c_2 \end{cases}$$

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

z_1 and z_2 are separated

$z_1' + J z_1 = c_1 \Rightarrow$ is an ODE
need to specify $z_1(0)$
for a unique solution

$N z_2' + z_1 = c_2 \Rightarrow$ is still a DAE BUT NOW
is solvable

Solution of DAE in the form

$$N z' + z = c$$

Nil potent matrix $N^p = 0$

$$z = c - N z' = c - N(c' - N z'') = c - N c' + N^2 z''$$
$$z' = c' - N z''$$
$$z'' = c'' - N^2 z'''$$

$\vdots$

$$z = c - N c' + N^2 c'' - N^3 c''' + \quad + (-1)^{p-1} N^{p-1} c^{(p-1)} + \underbrace{N^p}_{=0} z^p$$

$$= \sum_{k=0}^{p-1} (-1)^k N^k c^{(k)} \Rightarrow$$ No initial conditions are necessary!!

TRANSFORM LINEAR DAE TO AN ODE

$$N z' + z = c$$

$$\Rightarrow z = \sum_{k=0}^{p-1} (-1)^k N^k c^{(k)}$$

$$\Rightarrow z' = \sum_{k=0}^{p-1} (-1)^k N^k c^{(k+1)}$$ is an ODE

needs $z(0)$

$$\boxed{z(0) = \sum_{k=0}^{p-1} (-1)^k N^k c^{(k)}(0)}$$

INITIAL CONDITION determined by the DAE

p derivative of $Nz' + z = c$ are necessary to obtain an ODE $\Rightarrow$ DIFFERENTIAL INDEX p

TRANSFORM A DAE INTO AN ODE without THE USE OF KRONEKER NORMAL FORM.

$$Ay' + By = c$$

Using GAUSSIAN ELIMINATION ON A we can find permutations P and Q and a matrix L such that

$$LPAQ = \begin{pmatrix} I & W \\ 0 & 0 \end{pmatrix}$$

Example:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 3 \end{pmatrix}$$

$$
\begin{array}{c}1\\2\\3\\4\end{array}\begin{pmatrix}0&1&2&3\\1&1&0&0\\0&1&2&3\\1&2&2&3\end{pmatrix}
\longrightarrow
\begin{array}{c}2\\1\\3\\4\end{array}\begin{pmatrix}1&1&0&0\\0&1&2&3\\0&1&2&3\\\boxed{1}&2&2&3\end{pmatrix}
\xrightarrow[(4)-(1)]{}
\begin{array}{c}2\\1\\3\\4\end{array}\begin{pmatrix}1&\boxed{1}&0&0\\0&1&2&3\\0&1&2&3\\0&1&2&3\end{pmatrix}
$$

$$
\longrightarrow
\begin{array}{c}2\\1\\3\\4\end{array}\begin{pmatrix}1&0&-2&-3\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{pmatrix}
\longrightarrow
\left[\begin{array}{c|c}I&W\\\hline 0&0\end{array}\right]
$$

$$
P=\begin{pmatrix}0&1&&\\1&0&&\\&&1&\\&&&1\end{pmatrix}
$$

Example

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 3 & 1 & 1 \end{bmatrix} \end{array} \rightarrow \begin{array}{c|c} & \begin{matrix} 2 & 1 & 3 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 1 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ \boxed{2} & 0 & 2 & 3 \\ \boxed{3} & 0 & 1 & 1 \end{bmatrix} \end{array} \rightarrow \begin{array}{c|c} & \begin{matrix} 2 & 1 & 3 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 1 \\ (3)-2(1) \quad 3 \\ (4)-3(1) \quad 4 \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & 1 \end{bmatrix} \end{array}$$

$$\rightarrow \begin{array}{c|c} & \begin{matrix} 2 & 3 & 1 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 4 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} \quad -\frac{1}{2}(2) \quad \begin{array}{c|c} & \begin{matrix} 2 & 3 & 1 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 4 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 2 \end{bmatrix} \end{array} \rightarrow \begin{array}{c|c} & \begin{matrix} 2 & 3 & 1 & 4 \end{matrix} \\ \begin{matrix} 2 \\ 4 \\ 3 \\ 1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \end{array}$$

$$\begin{array}{c|cccc} & 2 & 3 & 1 & 4 \\ \hline 2 & 1 & 0 & 0 & 1/2 \\ 4 & 0 & 1 & 0 & -1/2 \\ 3 & 0 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 5/2 \end{array} \rightarrow \begin{array}{c|cccc} & 2 & 3 & 4 & 1 \\ \hline 2 & 1 & 0 & 1/2 & 0 \\ 4 & 0 & 1 & -1/2 & 0 \\ 3 & 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 5/2 & 0 \end{array} \ \frac{1}{3} \rightarrow \begin{array}{c|cccc} & 2 & 3 & 4 & 1 \\ \hline 2 & 1 & 0 & 1/2 & 0 \\ 4 & 0 & 1 & -1/2 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \boxed{5/2} & 0 \end{array}$$

$$\begin{array}{c|cccc} & 2 & 3 & 4 & 1 \\ \hline 2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{bmatrix} I & W \\ 0 & 0 \end{bmatrix} \qquad W = 0$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

GIVEN A exists P Q and L such that

$$LPAQ = \begin{pmatrix} I & W \\ 0 & 0 \end{pmatrix}$$

$$LPBQ = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

<u>Thus</u> $Ay' + By = c \quad LPAy' + LPBy = LPc$

$$Qz = y$$

$$\Rightarrow \quad LPAQz' + LPBQz = LPc$$

$$\begin{bmatrix} I & W \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

THUS

given $Ay' + By = c$
after some manipulation (GAUSS-LIKE)
the DAE is on the form

$$\begin{bmatrix} I & W \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' + \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{cases} z_1' + W z_2 + B_1 z_1 + B_2 z_2 = c_1 \\ B_3 z_1 + B_4 z_2 = c_2 \quad (*) \end{cases}$$

IF (A, B) is A REGULAR PENCIL B_3 and B_4 cannot be BOTH ZERO (EXERCIZE TRY TO PROVE IT)

$$(*) \Rightarrow B_3 z_1' + B_4 z_2' = c_2'$$

$$\begin{cases} z_1' + W z_2 + B_1 z_1 + B_2 z_2 = c_1 \\ B_3 z_1' + B_4 z_2' = c_2' \end{cases}$$

AFTER 1-derivation

$$\begin{bmatrix} I & W \\ B_3 & B_4 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}' + \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2' \end{pmatrix}$$

$\Downarrow$ Apply AGAIN GAUSSIAN ELIMINATION

$$\begin{bmatrix} I & \tilde{W} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}' + \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \\ \tilde{B}_3 & \tilde{B}_4 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

this matrix is bigger than

$$\begin{bmatrix} I & \tilde{W} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' + \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \\ \tilde{B}_3 & \tilde{B}_4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

derivative

AFTER A DERIVATION

$$\begin{bmatrix} I & \tilde{W} \\ \tilde{B}_3 & \tilde{B}_4 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' + \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2' \end{pmatrix}$$

Applying the procedure again and againg

$$Ix' + \Pi x = e$$

a regular ODE the number of derivation used is the INDEX of the DAE

DAE ⇒ ODE (LINEAR CASE)

$$\begin{cases} x' + y' + z' + 2x + z = t \\ x' + y' + z' + x + y = t^3 \\ \quad\quad\quad x + y + z = \sin(t) \end{cases}$$

$[A \;\; B \;\; C]$

$$\begin{array}{c|c} & \begin{matrix} x' & y' & z' & x & y & z & \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 1 & t \\ 1 & 1 & 1 & 1 & 1 & 0 & t^3 \\ 0 & 0 & 0 & 1 & 1 & 1 & \sin(t) \end{bmatrix} \end{array}$$

⊗ check if the pencil is regular $|A + \lambda B|$

$$\begin{vmatrix} 1+2\lambda & 1 & 1+\lambda \\ 1+\lambda & 1+\lambda & 1 \\ \lambda & \lambda & \lambda \end{vmatrix} = 2\lambda^3 \Rightarrow$$ Pencil is Regular

$$
\begin{array}{c c}
 & \begin{array}{cccccc c} x' & y' & z' & x & y & z & \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccccc c}
1 & 1 & 1 & 2 & 0 & 1 & t \\
\boxed{1} & 1 & 1 & 1 & 1 & 0 & t^3 \\
0 & 0 & 0 & 1 & 1 & 1 & \sin(t)
\end{array}\right]
\end{array}
\to
\begin{array}{c}
\begin{array}{cccccc c} x' & y' & z' & x & y & z & \end{array} \\
\left[\begin{array}{ccc|ccc c}
1 & 1 & 1 & 2 & 0 & 1 & t \\
\hline
0 & 0 & 0 & -1 & 1 & -1 & t^3 - t \\
0 & 0 & 0 & 1 & 1 & 1 & \sin(t)
\end{array}\right]
\end{array}
$$

Perform 1 derivation

$$
\begin{array}{c c}
 & \begin{array}{cccccc c} x' & y' & z' & x & y & z & \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccccc c}
1 & 1 & 1 & 2 & 0 & 1 & t \\
-1 & 1 & -1 & 0 & 0 & 0 & 3t^2 - 1 \\
1 & 1 & 1 & 0 & 0 & 0 & \cos t
\end{array}\right]
\end{array}
\to
\begin{array}{c c}
 & \begin{array}{cccccc c} x' & y' & z' & x & y & z & \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccccc c}
1 & 1 & 1 & 2 & 0 & 1 & t \\
0 & \boxed{2} & 0 & 2 & 0 & 1 & 3t^2 + t - 1 \\
0 & 0 & 0 & -2 & 0 & -1 & \cos t - t
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c c}
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\left[\begin{array}{cccccc c}
1 & 1 & 1 & 2 & 0 & 1 & t \\
0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\, t^2 + t/2 - 1/2 \\
0 & 0 & 0 & -2 & 0 & -1 & \cos t - t
\end{array}\right]
\end{array}
$$

$$\begin{array}{c|cccccc|c}
 & x' & y' & t' & x & y & z & \\
\hline
1 & 1 & 1 & 1 & 2 & 0 & 1 & t \\
2 & 0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\,t^2 + t/2 - 1/2 \\
3 & 0 & 0 & 0 & -2 & 0 & -1 & \cos t - t
\end{array}$$

$$\begin{array}{c|ccc|ccc|c}
1 & 1 & 0 & 1 & 1 & 0 & 1/2 & t/2 - 3/2\,t^2 + 1/2 \\
2 & 0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\,t^2 + t/2 - 1/2 \\
\hline
3 & 0 & 0 & 0 & -2 & 0 & -1 & \cos t - t
\end{array}$$

Apply a derivation (THE SECOND ONE)

$$\begin{array}{c|cccccc|c}
 & x' & y' & z' & x & y & z & \\
\hline
1 & 1 & 0 & 1 & 1 & 0 & 1/2 & t/2 - 3/2\,t^2 + 1/2 \\
2 & 0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\,t^2 + t/2 - 1/2 \\
3 & \boxed{-2} & 0 & -1 & 0 & 0 & 0 & -\sin t - 1
\end{array}$$

$$\begin{array}{cccccc|c}
0 & 0 & 1 & 2 & 0 & 1 & -\sin t + t - 3t^2
\end{array}$$

$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3
\end{array}
\begin{array}{c}
\begin{array}{cccccc} x' & y' & z' & x & y & z \end{array} \\
\left[\begin{array}{cccccc|c}
1 & 0 & \boxed{1} & 1 & 0 & 1/2 & t/2 - 3/2\, t^2 + 1/2 \\
0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\, t^2 + t/2 - 1/2 \\
0 & 0 & 1 & 2 & 0 & 1 & -\sin t + t - 3t^2
\end{array}\right]
\end{array}
$$

$$
\begin{array}{c}
1 \\ 2 \\ 3
\end{array}
\left[\begin{array}{ccc|ccc|c}
1 & 0 & 0 & -1 & 0 & -1/2 & -t/2 + 3/2\, t^2 + 1/2 + \sin t \\
0 & 1 & 0 & 1 & 0 & 1/2 & 3/2\, t^2 + t/2 - 1/2 \\
0 & 0 & 1 & 2 & 0 & 1 & -\sin t + t - 3t^2
\end{array}\right]
$$

we have derived 2 TIMES $\Rightarrow$ THE INDEX IS $\underline{2}$

$$
\begin{cases}
x' = x + \frac{z}{2} - \frac{t}{2} + \frac{3}{2} t^2 + \frac{1}{2} + \sin t \\
y' = -x - \frac{z}{2} + \frac{3}{2} t^2 + \frac{t}{2} - \frac{1}{2} \\
z' = -2x - z - \sin t + t - 3t^2
\end{cases}
$$

THE DIFFERENTIAL INDEX

A general DAE

$$F(y, y', t) = 0$$

IF we can rewrite as $y' = G(y, t)$

IF we can rewrite as $y' = G(y,t)$ then $F(y,y',t)$ is an ODE or a DAE of index-0

IF using

$$F(y, y', t) = 0$$

$$\frac{d}{dt} F(y, y', t) = 0 \quad \left[\frac{\partial F}{\partial y} y' + \frac{\partial F}{\partial y'} y'' + \frac{\partial F}{\partial t} = 0 \right]$$

we can extract $y' = G(y, t)$ then the DAE is an index-1 DAE

Example (INDEX -1 DAE)

$$y_1' + y_1 + y_2 = 1$$

(A) $$y_1^2 + y_2 + t = 0$$

using first derivative

$$y_1'' + y_1' + y_2' = 0$$

(B) $$2y_1 y_1' + y_2' + t = 0 \Rightarrow y_2' = -2y_1 y_1' - 1$$

using (A) on (B)

$$\begin{cases} y_1' = 1 - y_1 - y_2 \\ y_2' = -2y_1 y_1' - 1 = -2y_1(1 - y_1 - y_2) - 1 \end{cases}$$

ODE: $$\begin{cases} y_1' = 1 - y_1 - y_2 \\ y_2' = -2y_1(1 - y_1 - y_2) - 1 \end{cases}$$

In general given $F(y,y',t)=0$

using

$$\left.\begin{aligned} &F(y,y',t)=0 \\ &\frac{d}{dt}F(y,y',t)=0 \\ &\frac{d^2}{dt^2}F(y,y',t)=0 \\ &\vdots \\ &\left(\frac{d}{dt}\right)^p F(y,y',t)=0 \end{aligned}\right\} \Rightarrow y' = G(y,t) \quad (*)$$

The DAE is (AT MOST) of INDEX-p

if using less derivative cannot find $(*)$ the differential index is p