ODE FROM MECHANICAL PRINCIPLE

$$L(q,\dot q) = T(q,\dot q) - V(q)$$

$$T(q,\dot q) = \frac{1}{2}\,\dot q \cdot (M(q)\,\dot q)$$

L = Lagrangian
T = Kinetic Energy
V = Potential ENERGY

The physical trajectory satisfy least action principle
i.e. minimize

$$\mathcal{L} = \int_a^b L(q,\dot q)\,dt \quad \text{is minimized}$$

$q(a)$ and $q(b)$ fixed

$\Rightarrow$ Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = Q(q,t)$$

Q = generalized FORCE

$\Rightarrow$ THIS IS an ODE

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q}$$

This is valid when q are minimal coordinate, i.e., the minimum number of coordinates which determine the state of the system. item est = so that

In practical situation minimal coordinate can be impractical. Using reduntant coordinates give us a simpler representation of the mechanical (electric) system.

$L(q,\dot{q}) = T(q,\dot{q}) - V(q)$ L = Lagrangian

$T(q,\dot{q}) = \frac{1}{2}\dot{q}\cdot(M(q)\dot{q})$ T = kinetic Energy

V = Potential Energy

$\Phi(q,\dot{q})$ = extra constraints when q(t) are not minimal

minimize $\mathcal{L} = \int_a^b L(q,\dot{q})\,dt$ is minimized

q(a) and q(b) fixed

+ $\Phi(q,\dot{q}) = 0$

Least action + Constraints $\Rightarrow$ Constrained Euler-Lagrange

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot q}\right) - \dfrac{\partial L}{\partial q} + \left(\dfrac{\partial \phi}{\partial q}\right)^T \lambda = Q \\ \phi(q,t) = 0 \end{cases}$$

mat21.etsii.upm.es/mbs/bookPDFs/bookGjB.htm
(FREE BOOK of JALON + BAYO with more details)

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when $T(q,\dot q) = \frac{1}{2}\dot q^T M(q)\dot q$ then the DAE takes the form

$$\begin{cases} M(q)\ddot q + \dfrac{\partial \phi}{\partial q}^T(q,t)\lambda = Q(q,\dot q,t) = Q(q,t) + \dfrac{\partial L}{\partial q} - \dot M(q)\dot q \\ \phi(q,t) = 0. \end{cases}$$

UNKNOWN ARE $q(t)$ and $\lambda(t)$ $\Rightarrow$ $\lambda(t)$ is algebraic $\Rightarrow$ is a DAE

A wrong (but seems good) NUMERICAL SCHEME

$$\begin{cases} M(q)\ddot{q} + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) \\ \phi(q,t) = 0 \end{cases}$$

Use Taylor expansion for exact solution $q(t)$

$$q(t+\Delta t) = q(t) + \Delta t\, \dot{q}(t) + \frac{\Delta t^2}{2}\ddot{q}(t) + O(\Delta t^3)$$

$$\dot{q}(t+\Delta t) = \dot{q}(t) + \Delta t\, \ddot{q}(t) + O(\Delta t^2)$$

$$\left.\begin{array}{l} t_k = 0 + k\Delta t \\ q_k \simeq q(t_k) \quad \dot{q}_k \simeq \dot{q}(t_k) \end{array}\right\}$$ the approximate solution satisfy

$$q_{k+1} = q_k + \Delta t\, \dot{q}_k + \frac{\Delta t^2}{2}\ddot{q}_k$$

$$\dot{q}_{k+1} = \dot{q}_k + \Delta t\, \ddot{q}_k$$

FOR THE MOMENT NOT DEFINED

$$\begin{cases} M(q)\ddot{q} + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) \\ \phi(q,t) = 0 \end{cases}$$

$$\Rightarrow \ddot{q} = M(q)^{-1}\left[Q(q,\dot{q},t) - \frac{\partial \phi}{\partial q}(q,t)^T \lambda\right]$$

$$\ddot{q}_k = M(q_k)^{-1}\left[Q(q_k,\dot{q}_k,t_k) - \frac{\partial \phi}{\partial q}^T(q_k,t_k)\lambda_k\right]$$

we miss a way to compute $\lambda_k \approx \lambda(t_k) \Rightarrow$

use derivative of $\phi(q,t)$ to transform DAE to an ODE

$\phi(q(t),t) = 0$ along the exact solution

$$\frac{d}{dt}\phi(q(t),t) = 0 \Rightarrow \frac{\partial \phi}{\partial q}(q(t),t)\,\dot{q}(t) + \frac{\partial \phi}{\partial t} = 0$$

$$\frac{d^2}{dt^2}\phi(q(t),t) = 0 \Rightarrow \text{NEXT PAGE}$$

$$\left(\frac{d}{dt}\right)^2 \phi(q(t),t) =$$ is better to do component by component

$$\left(\frac{d}{dt}\right)^2 \phi_k(q(t),t) = \frac{\partial^2 \phi_k(q(t),t)}{\partial t^2} + 2\frac{\partial^2 \phi_k(q(t),t)}{\partial t \partial q}\dot{q}(t) + \dot{q}(t)^T \frac{\partial^2 \phi_k(q(t),t)}{\partial^2 q}\dot{q}(t) + \frac{\partial \phi_k(q(t),t)}{\partial q}\ddot{q}(t)$$

$$\left(\frac{d}{dt}\right)^2 \phi(q(t),t) = \frac{\partial \phi(q(t),t)}{\partial q}\ddot{q}(t) + c(q(t),\dot{q}(t),t)$$

found the missing equation for λ

$$\ddot{q} = M(q)^{-1}\left[Q(q,\dot{q},t) - \frac{\partial \phi}{\partial q}(q,t)^T \lambda\right]$$

Resume'

$$
\text{DAE:}\quad \begin{cases} M(q)\ddot{q} + \dfrac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) \\[2ex] \boxed{\phi(q,t) = 0} \end{cases} \qquad (A)
$$

$$
\left(\frac{d}{dt}\right)^2 \phi(q(t),t) = \boxed{\frac{\partial \phi}{\partial q}(q,t)\,\ddot{q}(t) + c(q(t),\dot{q}(t),t)}
$$

$$
\text{DAE:}\quad \begin{cases} M(q)\ddot{q} + \dfrac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) & (B.1) \\[2ex] \dfrac{\partial \phi}{\partial q}(q,t)\,\ddot{q} = -c(q,\dot{q},t) & (B.2) \end{cases}
$$

using: $\ddot{q} = M(q)^{-1}\left[Q(q,\dot{q},t) - \dfrac{\partial \phi}{\partial q}(q,t)^T \lambda\right]$ equation (B.2) can be transformed in an algebraic equation for $\lambda(t)$ (too complex) which derived produce an ODE for the original system (A) $\Rightarrow \lambda'' =$ [illegible]

Reduction to an ODE ⇒ can be done but is too complex, so use (B) directly

$$\text{DAE:} \begin{cases} M(q)\ddot{q} + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) & (B.1) \\ \frac{\partial \phi}{\partial q}(q,t)\ddot{q} = -c(q,\dot{q},t) & (B.2) \end{cases}$$

$$\Rightarrow \begin{bmatrix} M(q) & \frac{\partial \phi}{\partial q}(q,t)^T \\ \frac{\partial \phi}{\partial q}(q,t) & 0 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q(q,\dot{q},t) \\ -c(q,\dot{q},t) \end{pmatrix}$$

Solving this linear system permits to (FORMALLY) write

$$\ddot{q}(q,\dot{q},t)$$
$$\lambda(q,\dot{q},t)$$

Now we complete the (BAD) numerical scheme:

$t_k = a + k\Delta t$

$q_k \approx q(t_k) \quad \lambda_k \approx \lambda(t_k)$

$\dot{q}_k \approx \dot{q}(t_k)$

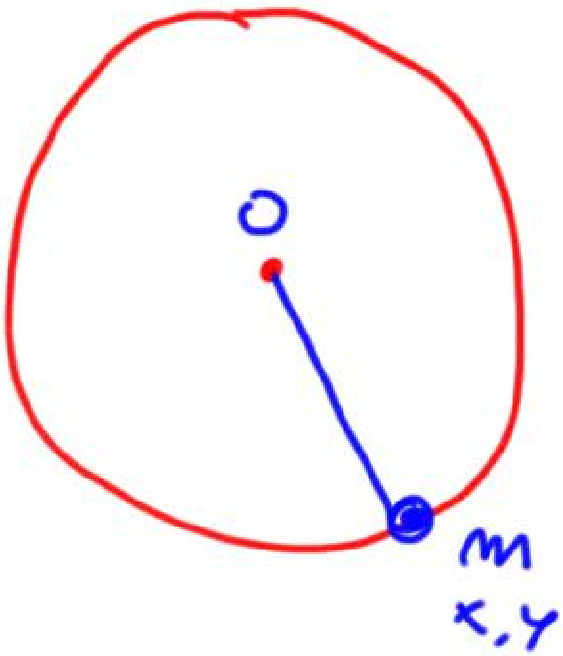
If we know $q_k, \dot{q}_k \Rightarrow$ compute $q_{k+1}, \dot{q}_{k+1}$

$$q_{k+1} = q_k + \Delta t\, \dot{q}_k + \frac{\Delta t^2}{2} \ddot{q}_k$$

$$\dot{q}_{k+1} = \dot{q}_k + \Delta t\, \ddot{q}_k$$

$$\begin{bmatrix} M(q_k) & \frac{\partial \phi}{\partial q}(q_k, t_k)^T \\ \frac{\partial \phi}{\partial q}(q_k, t_k) & 0 \end{bmatrix} \begin{pmatrix} \ddot{q}_k \\ \lambda_k \end{pmatrix} = \begin{pmatrix} Q(q_k, \dot{q}_k, t_k) \\ -c(q_k, \dot{q}_k, t_k) \end{pmatrix}$$

Example: Pendulum using cartesian coordinates



Kinetic energy: $\frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

Potential energy: $m g y$

Constraint: $\phi(x, y) = x^2 + y^2 - 1$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - m g y$$

using Euler-Lagrange equations with constraints $\Rightarrow$

$$\begin{cases} \boxed{\begin{array}{l} m \ddot{x} + 2 x \lambda = 0 \\ m \ddot{y} + 2 y \lambda = -m g \end{array}} & \text{ODE part} \\ \boxed{x^2 + y^2 = 1} & \text{constraint} \end{cases}$$

$$\begin{cases} \boxed{\begin{aligned} m\ddot{x} + 2x\lambda &= 0 \\ m\ddot{y} + 2y\lambda &= -mg \end{aligned}} \\ \boxed{x^2 + y^2 = 1} \end{cases}$$

ODE part $\Rightarrow$ $\ddot{x} = -\frac{2}{m}x\lambda$

$\ddot{y} = -g - \frac{2}{m}y\lambda$

constraint

First derivative of constraint

$$\frac{d}{dt}\phi(q,t) = \frac{d}{dt}(x^2+y^2-1) = 2x\dot{x} + 2y\dot{y}$$

Second derivative of constraint

$$\left(\frac{d}{dt}\right)^2\phi(q,t) = \frac{d}{dt}(2x\dot{x} + 2y\dot{y}) = \boxed{2x\ddot{x} + 2(\dot{x})^2 + 2y\ddot{y} + 2(\dot{y})^2}$$

$$\boxed{\begin{aligned} &= 2\left[-\frac{2}{m}x^2\lambda + (\dot{x})^2 - \frac{2}{m}y^2\lambda - yg + (\dot{y})^2\right] \\ &\overset{!}{=} -\frac{4}{m}(x^2+y^2)\lambda - 2yg + 2\left[(\dot{x})^2 + (\dot{y})^2\right] = 0 \end{aligned}}$$

only to show how to obtain an ODE

The new DAE is:

$$\begin{cases} m\ddot{x} + 2x\lambda = 0 \\ m\ddot{y} + 2y\lambda = -mg \\ 2x\ddot{x} + 2y\ddot{y} = -2\left((\dot{x})^2 + (\dot{y})^2\right) \end{cases}$$

$$\left[\begin{array}{cc|c} m & 0 & 2x \\ 0 & m & 2y \\ \hline 2x & 2y & 0 \end{array}\right] \left(\begin{array}{c} \ddot{x} \\ \ddot{y} \\ \hline \lambda \end{array}\right) = \left(\begin{array}{c} 0 \\ -mg \\ \hline -2\left((\dot{x})^2 + (\dot{y})^2\right) \end{array}\right)$$

For this simple example linear system can be solved explicitly

$$\ddot{x} = \frac{x(gy - s)}{r} \qquad r = x^2 + y^2$$

$$\ddot{y} = -\frac{gx^2 + ys}{r} \qquad s = (\dot{x})^2 + (\dot{y})^2$$

$$\lambda = -\frac{m(gy - s)}{2r}$$

The advancing scheme

$$x_{k+1} = x_k + \Delta t\, \dot{x}_k + \frac{\Delta t^2}{2} \ddot{x}_k$$

$$y_{k+1} = y_k + \Delta t\, \dot{y}_k + \frac{\Delta t^2}{2} \ddot{y}_k$$

$$\dot{x}_{k+1} = \dot{x}_k + \Delta t\, \ddot{x}_k$$

$$\dot{y}_{k+1} = \dot{y}_k + \Delta t\, \ddot{y}_k$$

$$\ddot{x}_k = \frac{x_k (g y_k - s_k)}{r_k} \qquad s_k = (\dot{x}_k)^2 + (\dot{y}_k)^2$$

$$\ddot{y}_k = - \frac{g x_k^2 + y_k s_k}{r_k} \qquad r_k = x_k^2 + y_k^2$$

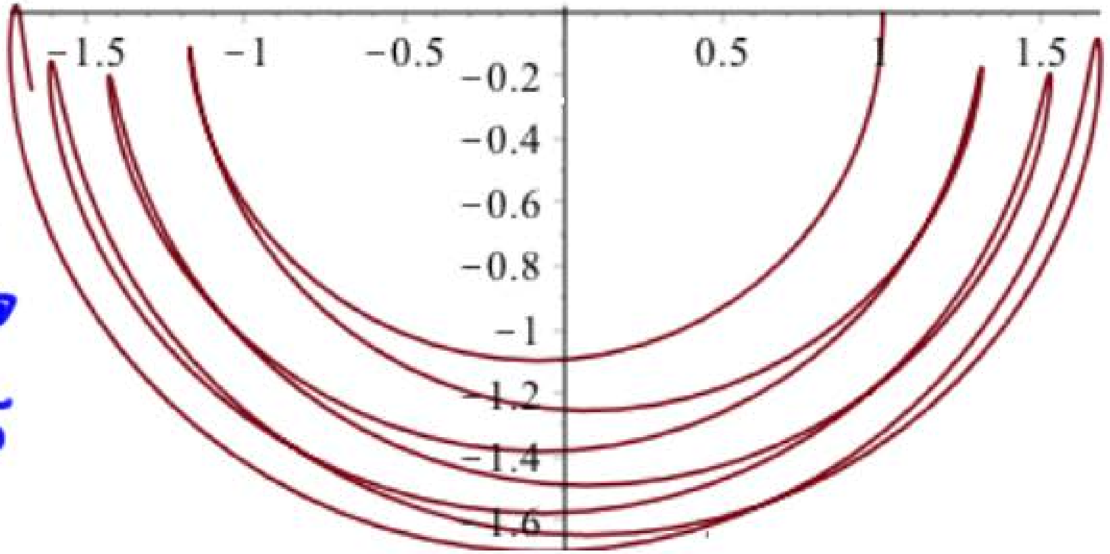
$x_0 \; y_0 \; \dot{x}_0 \; \dot{y}_0 \Rightarrow$ advance using the previous numerical scheme

Numerical solution for initial point

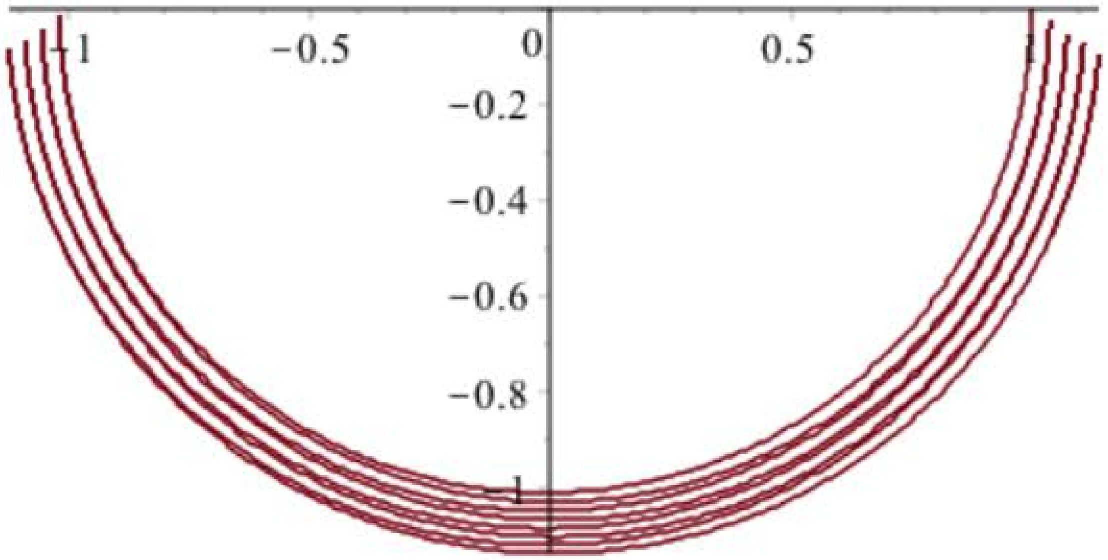
$x_0 = 1 \quad \dot{x}_0 = 0$

$y_0 = 0 \quad \dot{y}_0 = 0$

t range in $[0, 15]$ $\quad \Delta t = 0.025$



$\Delta t = 0,0025$



The numerical solution do not satisfy the constraint $x^2 + y^2 = 1$

The error is called

DRIFT

WHY THERE IS THE DRIFT ?

The original constraint

$$\phi(q,t)$$

was substituted by the new equations

$$\left(\frac{d}{dt}\right)^2 \phi(q(t),t) = \frac{\partial\phi}{\partial q}(q,t)\,\ddot{q} + c(q,\dot{q},t)$$

but if you use a new constraint

$$\tilde{\phi}(q,t) = \phi(q,t) + c_1 + c_2 t$$

this contraint has the same second derivative of the original constraint

$$\left(\frac{d}{dt}\right)^2 \tilde{\phi}(q,t) = \left(\frac{d}{dt}\right)^2 \phi(q,t)$$

The transformed DAE

$$\begin{cases} M(q)\ddot{q} + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) \\ \frac{\partial \phi}{\partial q}(q,t)\ddot{q} \qquad\qquad = -c(q,\dot{q},t) \end{cases}$$

Contain the solution of all these DAE

$$\begin{cases} M(q)\ddot{q} + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,\dot{q},t) \\ \phi(q,t) + c_1 + c_2 t = 0 \end{cases}$$

The constants c_1 and c_2 are determined by the initial conditions. For example

$$\phi(q(0),0) + c_1 \qquad \frac{\partial \phi}{\partial q}(q(0),0)\,\dot{q}(0) + \frac{\partial \phi}{\partial t}(q(0),0) + c_2$$