BAUMGARTE STABILIZATION

A second order index-3 DAE

$z = q'$ reduce to 1st order

$$\begin{cases} M(q)\,q'' + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,q') \\ \phi(q,t) = 0 \end{cases}$$

$M(q)$ not singular

check that DAE is of index 3

1 - derivation $\quad \frac{\partial \phi}{\partial q} q' + \frac{\partial \phi}{\partial t} = 0$

obtain λ

2 - derivation $\quad \frac{\partial^2 \phi}{\partial q^2}(q')^2 + \frac{\partial \phi}{\partial q} q'' + \frac{\partial \phi}{\partial q \partial t} q' + \frac{\partial^2 \phi}{\partial t^2} = 0$

3 - derivation $\quad$ OBTAIN AN EQUATION with λ'

INDEX 3 ?

$$\begin{cases} M(q)\,q'' + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,q') \\ \phi(q,t) = 0 \end{cases}$$

$M(q)$ NOT SINGULAR!

Reduce to 1st order

$$\begin{cases} q' = z \\ M(q)\,z' + \frac{\partial \phi}{\partial q}(q,t)^T \lambda = Q(q,z) \\ \phi(q,t) = 0 \end{cases}$$

$\Rightarrow z' = M(q)^{-1}(\;\;)$

1 - derivation $\quad \frac{\partial \phi}{\partial q} q' + \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial q} z + \frac{\partial \phi}{\partial t} = 0$

2 - derivation $\quad \frac{\partial^2 \phi}{\partial q^2} z z + \frac{\partial \phi}{\partial q} z' + \frac{\partial^2 \phi}{\partial q \partial t} z + \frac{\partial^2 \phi}{\partial t^2} = 0$

3 - derivation $\quad$ AFTER SOME MANIPULATION $\lambda' = \dots$

LAST WEEK (NOT A) SOLUTION

$$\begin{cases} M(q)\,q'' + \dfrac{\partial \phi}{\partial q}(q,t)\,\lambda = Q(q,q') \\ \boxed{\phi(q,t) = 0} \end{cases}$$

substitute this with

$$\frac{d^2}{dt^2}\phi(q,t) = \frac{\partial \phi}{\partial q}(q,\lambda)\,q'' + \boxed{}$$

obtaining an INDEX-1 DAE

$\Downarrow$

solve the linear system $\begin{bmatrix} M(q) & \frac{\partial \phi}{\partial q}^T \\ \frac{\partial \phi}{\partial q} & 0 \end{bmatrix} \begin{pmatrix} q'' \\ \lambda \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$

LAST WEEK (NOT A) SOLUTION (2)

Use Taylor series of exact solution

$$q(t+\Delta t) = q(t) + \Delta t\, q'(t) + \frac{\Delta t^2}{2} q''(t) + O(\Delta t^3)$$

$$q'(t+\Delta t) = q'(t) + \Delta t\, q''(t) + O(\Delta t^2)$$

Use the linear system

$$\begin{bmatrix} M(q) & \frac{\partial \phi}{\partial q}^T \\ \frac{\partial \phi}{\partial q} & 0 \end{bmatrix} \begin{pmatrix} q'' \\ \lambda \end{pmatrix} = \begin{pmatrix} r_1(q,q') \\ r_2(q,q') \end{pmatrix}$$

to find $q''(t)$ and build the advancing scheme

$$q_{k+1} = q_k + \Delta t\, q'_k + \frac{\Delta t^2}{2} q''_k$$

$$q'_{k+1} = q'_k + \Delta t\, q''_k$$

WHY DO NOT WORK?

the constraint is satisfied by the exact solution

$$\phi(q(t),t)=0 \Rightarrow \boxed{\frac{d^2}{dt^2}\phi(q(t),t)=0}$$

Equation used instead of $\phi(q,t)=0$

but $$\psi(q,t)=\phi(q,t)+c_1+c_2 t$$

$$\frac{d^2}{dt^2}\psi(q,t)=\frac{d^2}{dt^2}\phi(q,t)$$

thus the numerical scheme approximate also the problem with constraint $\psi(q,t)$

WHY DO NOT WORK? (2)

Depending on the initial value $q(0)$ $q'(0)$ the constant c_1 and c_2 are determined

$$\psi(q(0), 0) = \phi(q(0), 0) + c_1 + c_2 0 = 0$$

$$\frac{d}{dt}\psi(q(0), 0) = \frac{\partial \psi}{\partial q} q'(0) + \frac{\partial \psi}{\partial t} = \frac{\partial \phi}{\partial q} q'(0) + \frac{\partial \phi}{\partial t} + c_2 = 0$$

Thus PERFECT initial condition select $c_1 = c_2 = 0$ the right constraint. BUT local error of numerical method at EACH step introduce a SMALL ERROR which means that $\phi(q_n, t_n) \neq 0$ or

$$\phi(q_n, t_n) + c_1 + c_2 t_n = 0$$

so c_1 and c_2 becomes $\neq 0$ $\Rightarrow$ the error grows linearly

BAUMGARTE SOLUTION

$z(t) = \phi(q(t), t)$

$$\begin{cases} M(q)q'' + \dfrac{\partial \phi}{\partial q}\lambda = Q(q,q') \\ \underbrace{\phi(q,t) = 0}_{z(t)=0} \end{cases} \Rightarrow \begin{cases} M(q)q'' + \dfrac{\partial \phi}{\partial q}\lambda = Q(q,q') \\ z''(t) = 0 \end{cases}$$

also $z(t) + c_1 + c_2 t$ satisfy this equation

it should be better to satisfy $z(t) = 0$ $z'(t) = 0$ $z''(t) = 0$

the solution of

$$z''(t) = 0 \Rightarrow z(t) = c_1 + c_2 t$$

if consider the ODE $\quad z'' + a z' + b z = 0$

$z(t) = 0$ is a solution BUT ....

.... BUT

the general solution of

$$z'' + a z' + b z = 0 \quad (*)$$

is

$$z(t) = c_1 e^{\frac{\Delta - a}{2} t} + c_2 e^{-\frac{\Delta + a}{2} t} \qquad \Delta = \sqrt{a^2 - 4b}$$

if the real part of $-\frac{a}{2} \pm \frac{\Delta}{2}$ is negative
the solution asymptotically goes to 0 as $t \to \infty$
thus no matter the initial value of $z(t)$ because $z(t) \to 0$ $t \to \infty$

Usually $(*)$ is written as

$$z'' + 2\omega\eta z' + \omega^2 z = 0$$

$$z(t) = c_1 e^{\omega t\left(-\eta + \sqrt{\eta^2 - 1}\right)} + c_1 e^{\omega t\left(-\eta - \sqrt{\eta^2 - 1}\right)}$$

thus if $0 \leq \eta < 1$ $\sqrt{\eta^2 - 1}$ is pure unimaginary
and real part of exponent is negative

PRACTICA 2nd ORDER DUMPING TO ODE

$$z'' + 2\omega\eta z' + \omega^2 z = 0$$

$$z(t) = c_1 e^{\omega t\left(-\eta + \sqrt{\eta^2 - 1}\right)} + c_1 e^{\omega t\left(-\eta - \sqrt{\eta^2 - 1}\right)}$$

when $0 < \eta \leq 1$ the exponential is of the form

$$z(t) = e^{-\eta\omega t}\left[c_1 e^{j\sqrt{1-\eta^2}\,\omega t} + c_2 e^{-j\sqrt{1-\eta^2}\,\omega t}\right]$$

$$= \underbrace{e^{-\eta\omega t}}_{\text{term going to } 0}\underbrace{\left[\tilde{c}_1 \cos\sqrt{1-\eta^2}\,\omega t + \tilde{c}_2 \sin\sqrt{1-\eta^2}\,\omega t\right]}_{\text{obscillating but limited term}}$$

SPECIAL CASE WHEN $\eta = 1$

$$z(t) = e^{-\eta\omega t}\left[c_1 e^{j\sqrt{1-\eta^2}\,\omega t} + c_2 e^{-j\sqrt{1-\eta^2}\,\omega t}\right]$$

when $\eta = 1$ $\quad 1-\eta^2 = 0 \quad \cos 0 = 1 \quad \sin 0 = 0$

$z(t) = e^{-\omega t} c_1 \Rightarrow$ critical damping
the solution goes
monotonically to 0

Trust using

$z'' + 2\eta\omega z' + \omega^2 z = 0$ where $z = \phi(y, t)$

if $z(t)$ is $\neq 0 \Rightarrow$ the discretization of ODE
push $z \to 0$

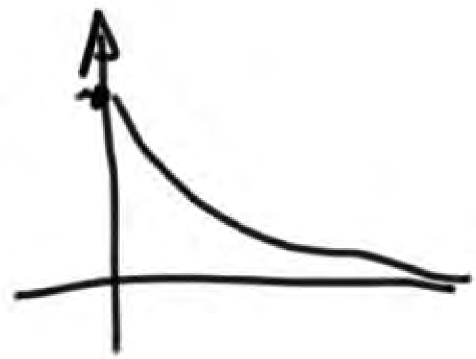
- ω is a parameter which dump the error
  ω large $\Rightarrow$ error fastly dumped
- thus choose ω as large as possible to dump the error in order to push the error near 0.

PROBLEM ω cannot be choosen as large as one want.

Why?

Example with a simple ODE

$$\begin{cases} y' + \omega y = 0 \\ y(0) = 1 \end{cases} \Rightarrow y(t) = e^{-\omega t}$$



Ex 1

Numerical method (Explicit Euler)

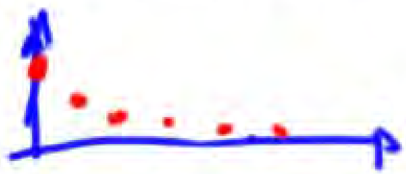
$$\begin{cases} y' + \omega y = 0 \\ y(0) = 1 \end{cases} \qquad y(t) = e^{-\omega t}$$

$$\frac{y_{k+1} - y_k}{\Delta t} + \omega y_k = 0 \Rightarrow y_{k+1} = y_k - \Delta t \, \omega y_k$$

$$y_0 = 1$$

$$y_{k+1} = (1 - \Delta t \, \omega) y_k = (1 - \Delta t \, \omega)^{k+1}$$

when $\Delta t \, \omega < 1$



OK

when $\Delta t \, \omega = 1$

OK

when $2 > \Delta t \, \omega > 1$

mildly OK

$$\frac{y_{k+1}-y_k}{\Delta t} + \omega y_k = 0 \Rightarrow y_{k+1} = y_k - \Delta t \omega y_k$$

$y_0 = 1$

$$y_{k+1} = (1-\Delta t\, \omega)\, y_k = (1-\Delta t\, \omega)^{k+1}$$

when $\Delta t\, \omega = 2$

No dumping

OK → No

when $\Delta t\, \omega > 2$

the solution explode

$\Rightarrow \Delta t < \frac{2}{\omega}$ to dump the numerical solution, if ω is LARGE Δt must be SMALL

TO AVOID TIME STEP LIMITATION USE FOR EXAMPLE IMPLICIT EULER

$$\frac{y_{n+1} - y_n}{\Delta t} + \omega\, y_{n+1} = 0 \qquad (1+\Delta t\,\omega)\, y_{n+1} = y_n$$

$$\begin{cases} y_0 = 1 \\ y_k = \dfrac{1}{(1+\Delta t\,\omega)^k} \end{cases} \Rightarrow$$

No Δt limitation to damp numerical solution.

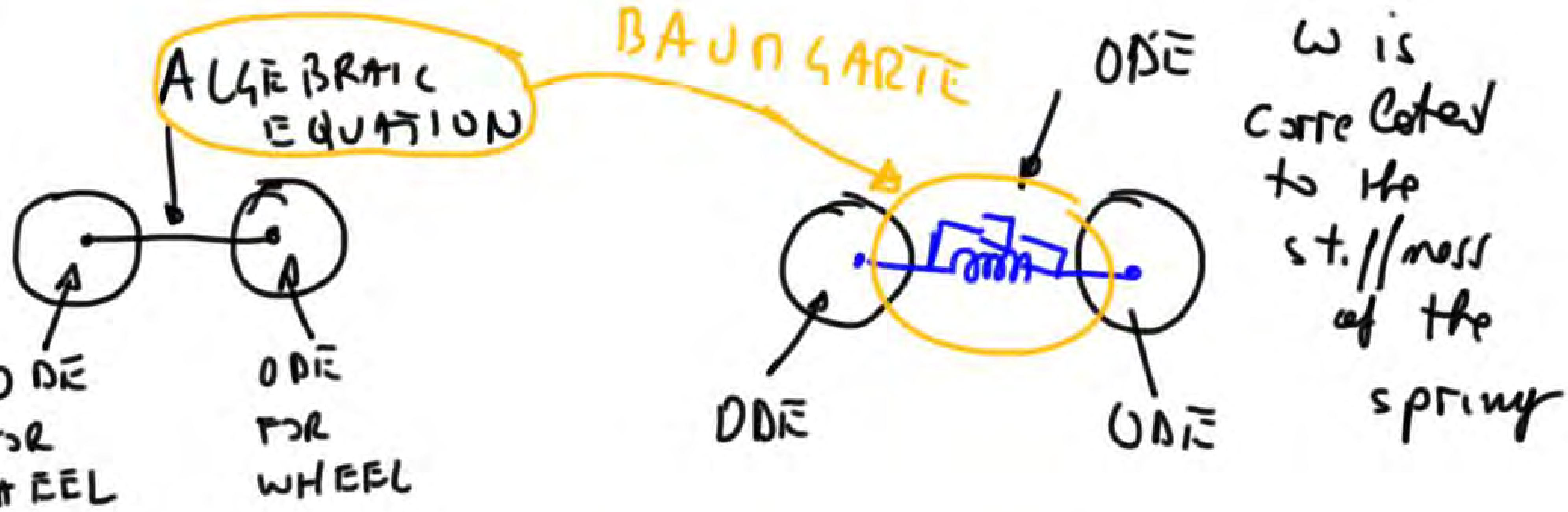
"PRACTICAL DEFINITION"

An ODE is stiff when the step limitation of the numerical scheme is NOT determined by the ACCURACY but by the STABILITY of numerical scheme.

BAUMGARTE with large ω produce stiff ODE

FINAL CONSIDERATION with BAUNGARTE IDEA

A DAE FOR A MULTIBODY SYSTEM



$\Rightarrow$ CHANGE THE PHYSICS OF THE MODEL

SPURIOUS OBSCILLATION

NON PHYSICS RESONANCE ....

Index 1 semi explicit DAE

$$\begin{cases} x' = f(x, y, t) \\ 0 = g(x, y, t) \end{cases}$$

$\Rightarrow$ Index 1 means that y can be determined from $g(x, y, t) = 0$ using IMPLICIT FUNCTION THEOREM.

$\Rightarrow \frac{\partial g}{\partial y}(x, y, t)$ is NOT singular

FORMALLY solve $g(x, y, t) = 0 \Rightarrow y(x, t)$

$g(x, y(x, t), t) = 0$

using $y(x, t)$ obtain an ODE

$$x' = f(x, y(x, t), t) = h(x, t) \qquad h(x, t) = f(x, y(x, t), t)$$

Example using Explicit Euler

$$\begin{cases} x' = f(x, y, t) \\ 0 = g(x, y, t) \end{cases}$$

$$\frac{x_{k+1} - x_k}{\Delta t} = f(x_k, y_k, t_n) \Rightarrow x_{k+1} = x_n + \Delta t\, f(x_n, y_n, t_n)$$

y_k is the solution of $g(x_k, y, t_n) = 0$

Use for example Newton or other numerical scheme.

BAD IDEA USE BAUMGARTE

$$\begin{cases} x' = f(x,y,t) \\ 0 = \boxed{g(x,y,t)} \end{cases}$$

$z(t) \Rightarrow z'(t) + 0\, z(t) = 0$

$$\begin{cases} x' = f(x,y,t) \\ \frac{d}{dt}\underbrace{g(x(t),y(t),t)}_{z(t)} + \omega \underbrace{g(x,y,t)}_{z(t)} = 0 \end{cases}$$

$$z' + \omega z = 0 \Rightarrow z(t) = z(0)\, e^{-\omega t}$$

$$\frac{d}{dt} g(x(t),y(t),t) = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial t}$$

$$= \frac{\partial g}{\partial x} f + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial t}$$

$$\left(\frac{\partial g}{\partial y}\right) y' + \frac{\partial g}{\partial x} f + \frac{\partial g}{\partial t} + \omega\, g(x,y,t) = 0$$ $\Rightarrow$ the constraint stabilized ODE of

Index-2 semi explicit DAE

$$\begin{cases} x' = f(x, y, t) \\ 0 = g(x, t) \end{cases}$$ ← cannot solve for $y(t)$!!!

Use BAUMGARTE

$$\frac{d}{dt} g(x(t), t) = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}(x, t) f(x, y, t) + \frac{\partial g}{\partial t}(x, t) = 0$$

Index-2 $\Rightarrow$ $\frac{\partial g}{\partial x}(x, t) \frac{\partial f}{\partial y}(x, y, t)$ is NOT singular

$$h(x, y, t) = \frac{\partial g}{\partial x}(x, t) f(x, y, t) + \frac{\partial g}{\partial t}(x, t)$$

$$\begin{cases} x' = f(x, y, t) \\ 0 = h(x, y, t) \end{cases}$$ is an Index-1 DAE

$$\begin{cases} x' = f(x, y, t) \\ 0 = h(x, y, t) \end{cases}$$

using explicit Euler
assuming y solved from $h(x, y, t)$

$$x_{k+1} = x_k + \Delta t \, f(x_k, y_k, t_k)$$

$$h(x_k, y, t_k) = 0 \Rightarrow$$ DO NOT WORK
we have a drift
of the original
constraint $g(x, t) = 0$

$\Rightarrow$ Use a "STABILIZED $h(x, y, t)$"

Index -2 semi explicit DAE

$$\begin{cases} x' = f(x,y,t) \\ 0 = g(x,t) \end{cases}$$

Index 1 stable DAE

$$\begin{cases} x' = f(x,y,t) \\ 0 = h(x,y,t) \end{cases}$$

STABILIZED ODE

$$\frac{d}{dt} g(x(t),t) + \omega\, g(x(t),t) = 0$$

$$\frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial t} + \omega\, g(x,t) = 0$$

$$\frac{\partial g}{\partial x}(x,t)\, f(x,y,t) + \frac{\partial g}{\partial t}(x,t) + \omega\, g(x,t) = 0$$

$$h(x,y,t) = \frac{\partial g}{\partial x} f(x,y,t) + \frac{\partial g}{\partial t} + \omega\, g(x,t)$$

Index -2 semi explicit DAE

$$\begin{cases} x' = f(x, y, t) \\ 0 = g(x, t) \end{cases}$$

NOT RECCOMENDED (IT WORK BUT MORE COSTLY)

$$\frac{d^2}{dt} g(x(t), t) + 2\omega\eta \frac{d}{dt} g(x(t), t) + \omega^2 g(x(t), t) = 0$$

here you obtain an ODE for Y

$$\begin{cases} x' = f(x, y, t) \\ \boxed{x} \end{cases}$$

is a regular ODE stabilized with BAUMGARTE IDEA