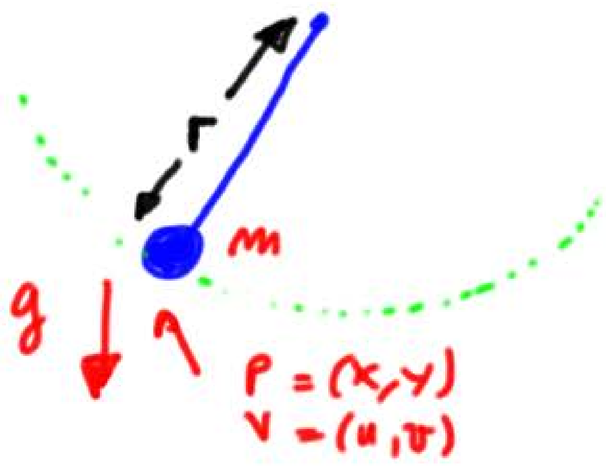
COORDINATE PARTITION FOR DAE

EXAMPLE (PENDULUM)



$$m x'' + 2x\lambda = 0$$

$$m y'' + 2y\lambda = -mg$$

$$x^2 + y^2 = r^2$$

setting $\mu = -2\lambda/m$

$$x'' = \mu x$$

$$y'' = \mu y - g$$

$$x^2 + y^2 = r$$

setting $u = x'$ $v = y'$ transform to 1st order ODE

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ x^2 + y^2 = r^2 \end{cases}$$

DAE FOR THE PENDULUM

COMPUTE THE DIFFERENTIAL INDEX

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ x^2 + y^2 = r^2 \end{cases}$$

$g(x,y) = x^2 + y^2 - r^2$ (constraint)

$$\frac{d}{dt} g(x(t),y(t)) = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' = 2 x x' + 2 y y' = \underbrace{2xu + 2yv}_{\text{algebraic}}$$

$$\left(\frac{d}{dt}\right)^2 g(x,y) = 2x'u + 2xu' + 2y'v + 2y v'$$

$$= 2u^2 + 2\mu x^2 + 2v^2 + 2\mu y^2 - 2yg$$

$$= 2(u^2 + v^2) + 2\mu (x^2 + y^2) - 2yg \leftarrow \text{algebraic}$$

$$\left(\frac{d}{dt}\right)^3 g(x,y) = 4(uu' + vv') + 2\mu'(x^2+y^2) + 4\mu(xx' + yy') - 2y'g$$

$$= 4\mu(xu + yv) - 4vg + 2\mu'(x^2+y^2) + 4\mu(xu+yv) - 2vg$$

$$= 8\mu(xu + yv) - 6vg + 2\mu'(x^2+y^2)$$

$$\Rightarrow \mu' = \frac{3vg - 4\mu(xu+yv)}{r^2} \Rightarrow \text{DAE is of INDEX 3}$$

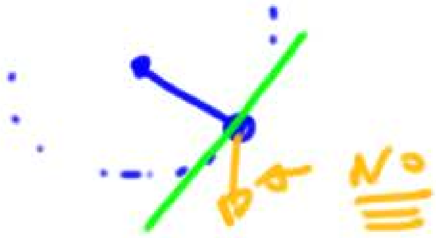
STABILIZATION USING BAUMGARTE

a) $g(x,y) = x^2 + y^2 - r^2$

b) $\frac{d}{dt} g(x,y) = 2xu + 2yv$ ← HIDDEN CONSTRAINT

c) $\left(\frac{d}{dt}\right)^2 g(x,y) = 2(u^2+v^2) + 2\lambda(\underbrace{x^2+y^2}_{r^2}) - 2yg$

(a) ⇒ BE ON THE CIRCLE



(b) ⇒ $2\binom{x}{y}\cdot\binom{u}{v}$ ⇒ velocity ortogonal to the ray

(c) ⇒ $\lambda = \frac{yg-(u^2+v^2)}{r^2}$ λ is proportional to centripetal force

$$\left(\frac{d}{dt}\right)^2 g(x,y) + 2\eta\omega \frac{d}{dt} g(x,y) + \omega^2 g(x,y) = 0$$

$$\boxed{0 \leq \eta \leq 1 \quad \omega > 0}$$

$$\Rightarrow \lambda = \lambda(x,y,u,v)$$

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ x^2 + y^2 = r^2 \end{cases}$$

INDEX-3 DAE

USING BAUMGARTE $\Longrightarrow$

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ \underbrace{\mu = \mu(x, y, u, v)}_{\Downarrow} \end{cases} \Longleftarrow$$

INDEX-1 DAE

Usually constraint cannot be explicitly solved

$$\left(\frac{d}{dt}\right)^2 g + 2\eta\omega \left(\frac{d}{dt}\right) g + \omega^2 g = 2(u^2 + v^2) + 2\mu(x^2 + y^2) - 2yg$$
$$4\eta\omega(xu + yv) + \omega^2(x^2 + y^2 - r^2) = 0$$

$$\Longrightarrow \mu = \frac{yg - (u^2 + v) - 2\eta\omega(xu + yv) - \frac{\omega^2}{2}(x^2 + y^2 - r^2)}{x^2 + y^2}$$

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu(x, y, u, v)\, x \\ v' = \mu(x, y, u, v)\, y - g \end{cases} \Longleftarrow \text{ODE}$$

COORDINATE PARTITION APPROACH

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - y \\ \boxed{x^2 + y^2 = r^2} \end{cases}$$

PARTITION (x,y) in TWO SET

DEPENDENT

INDEPENDENT COORDINATES

y dependent $\Rightarrow$ $y(x, u, v)$

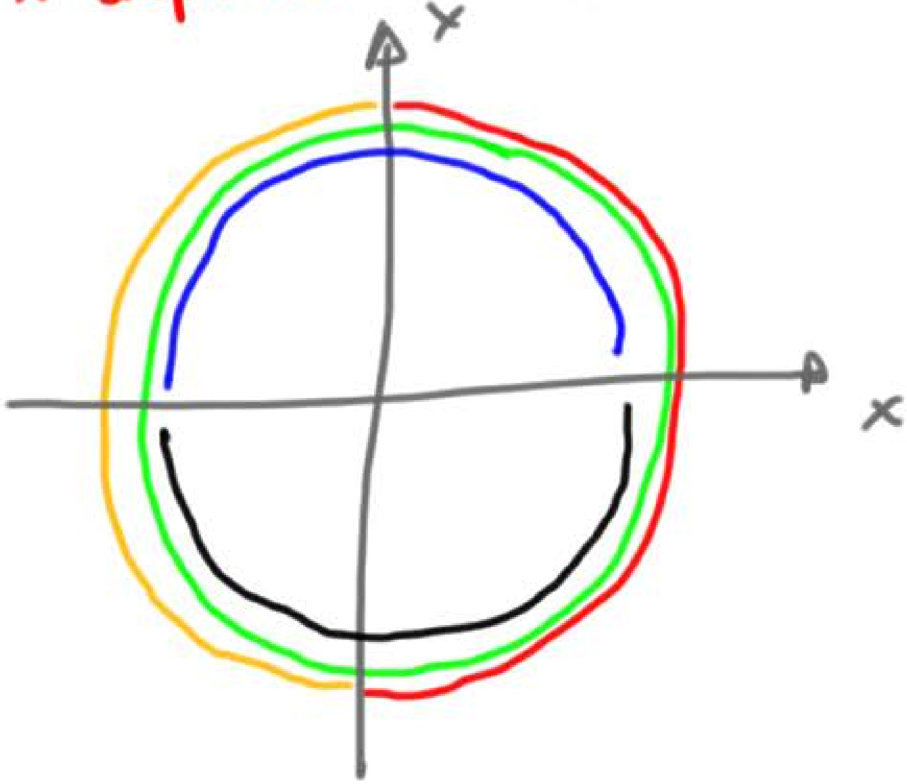
x dependent $\Rightarrow$ $x(y, u, v)$

$$x(y) = +\sqrt{r^2 - y^2}$$

$$x(y) = -\sqrt{r^2 - y^2}$$

$$y(x) = +\sqrt{r^2 - x^2}$$

$$y(x) = -\sqrt{r^2 - x^2}$$



NOW USE MAP

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ \boxed{x^2 + y^2 = r^2} \end{cases}$$

$$y(x) = -\sqrt{r^2 - x^2}$$

$$\Rightarrow x^2 + y(x)^2 = x^2 + r^2 - x^2 = r^2$$

using this map $y(t) \underset{\text{def}}{=} y(x(t))$

$$x'(t) = u(t)$$

$$y'(t) = \frac{d}{dt} y(t) = \frac{d}{dt} y(x(t)) = \frac{\partial y}{\partial x}(x(t))\, x'(t) = v(t)$$

$$\frac{\partial y}{\partial x} = -\frac{\partial}{\partial x}\sqrt{r^2 - x^2} = \frac{x}{\sqrt{r^2 - x^2}}$$

$$\boxed{\frac{x}{\sqrt{r^2 - x^2}}\, x' = v}$$

$$\Rightarrow \begin{cases} x' = u \\ x x' = \sqrt{r^2 - x^2}\, v \Rightarrow x u = \sqrt{r^2 - x^2}\, v \\ u' = \mu x \\ v' = \mu y(x) - g \end{cases}$$

$$\Updownarrow$$

$$v = v(x, u) = \frac{x u}{\sqrt{r^2 - x^2}}$$

$$\begin{cases} x' = u \\ y' = v \\ u' = \lambda x \\ v' = \lambda y - g \\ \boxed{x^2 + y^2 = r^2} \end{cases} + \quad y(x) = -\sqrt{r^2 - x^2} \quad \Rightarrow \quad \begin{cases} x' = u \\ u' = \lambda x \\ \boxed{v' = \lambda\, y(x) - g} \\ y(x) = -\sqrt{r^2 - x^2} \\ v(x, u) = \dfrac{xu}{\sqrt{r^2 - x^2}} \end{cases}$$

$$v' = \frac{d}{dt} v(x,u) = \frac{\partial v}{\partial x} x' + \frac{\partial v}{\partial u} u' = \frac{u r^3}{(r^2 - x^2)^{3/2}} x' + \frac{x}{\sqrt{r^2 - x^2}} u'$$

$$= \frac{r^2 u^2}{(r^2 - x^2)^{\frac{3}{2}}} + \frac{\lambda x^2}{\sqrt{r^2 - x^2}}$$

$$v' = \lambda\, y(x) - g \Rightarrow \lambda = \frac{g (r^2 - x^2)^{3/2} + r^2 u^2}{r^2 (x^2 - r^2)} = \lambda(x, u)$$

$$\Rightarrow \begin{cases} x' = u \\ u' = \lambda(x,u)\, x \end{cases} \qquad \lambda(x,u)$$

$\Leftarrow$ This ODE is valid for

$-1 < x < +1$

$-1 < y < 0$

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ \boxed{x^2 + y^2 = r^2} \end{cases}$$

$$x(y) = +\sqrt{r^2 - y^2}$$

$$x(y)^2 + y^2 = r^2 - y^2 + y^2 = r^2$$

using this map $x(t) \overset{\text{def}}{=} x(y(t))$

$$x'(t) = \frac{d}{dt} x(y(t)) = \frac{\partial x}{\partial y} y' = \frac{-y}{\sqrt{r^2 - y^2}} v = \frac{-yv}{\sqrt{r^2 - y^2}}$$

$$\Rightarrow u = -\frac{yv}{\sqrt{r^2 - y^2}} \Rightarrow u(y, v) = -\frac{yv}{\sqrt{r^2 - y^2}}$$

$$\begin{cases} y' = v \\ \frac{d}{dt} u(y, v) = \mu x(y) \\ v' = \mu y - g \\ x(y) = \sqrt{r^2 - y^2} \end{cases}$$

$$\begin{cases} x' = u \\ y' = v \\ u' = \mu x \\ v' = \mu y - g \\ \boxed{x^2 + y^2 = r^2} \end{cases} + \quad y(x) = \sqrt{r^2 - x^2}$$

$$\Rightarrow \begin{cases} y' = v \qquad u(y,v) = -\dfrac{y v}{\sqrt{r^2 - y^2}} \\ \dfrac{d}{dt} u(y,v) = \mu \, x(y) \\ v' = \mu y - g \\ x(y) = \sqrt{r^2 - y^2} \end{cases}$$

$$u' = \frac{d}{dt} u(y,v) = \frac{\partial u}{\partial y} y' + \frac{\partial u}{\partial v} v' = -\frac{v r^2 \, y'}{(r^2 - y^2)^{3/2}} - \frac{y}{\sqrt{r^2 - y^2}} v'$$

$$u' = \mu x \quad \Rightarrow \quad \mu = \frac{v^2}{y^2 - r^2} - \frac{g y}{r^2} \quad = \mu(y, v)$$

$$\Rightarrow \begin{cases} y' = v \\ v' = \mu(y,v) \, y - g \end{cases}$$

$\Leftarrow$ This ODE is valid for

$-1 < x < 1$

$0 < y < 1$

MINIMAL COORDINATE APPROACH

$$\begin{cases} x' = u \\ y' = v \\ u' = \lambda x \\ v' = \lambda y - g \\ \boxed{x^2 + y^2 = r^2} \end{cases}$$

New coordinate θ and x and y are function of θ

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$x(t) = r\cos\theta(t) \qquad x'(t) = -r\sin\theta(t)\,\theta'(t) = u$$

$$y(t) = r\sin\theta(t) \qquad y'(t) = r\cos\theta(t)\,\theta'(t) = v$$

$$\begin{cases} -r\sin\theta\,\theta' = u \\ r\cos\theta\,\theta' = v \\ u' = \lambda r\cos\theta \\ v' = \lambda r\sin\theta - g \end{cases}$$

$$\begin{matrix}\sin\theta \\ \cos\theta\end{matrix} \Rightarrow r(\sin\theta^2 + \cos\theta^2)\,\theta' = v\cos\theta - u\sin\theta$$

$$\theta' = \frac{1}{r}(v\cos\theta - u\sin\theta)$$

$$u' = \frac{d}{dt}(-r\sin\theta\,\theta') = -r(\cos\theta\,(\theta')^2 + \sin\theta\,\theta'')$$

$$v' = \frac{d}{dt}(r\cos\theta\,\theta') = r(-\sin\theta\,(\theta')^2 + \cos\theta\,\theta'')$$

$$\begin{cases} -r\sin\theta\,\theta' = u \\ r\cos\theta\,\theta' = v \\ u' = \mu\, r\cos\theta \\ v' = \mu\, r\sin\theta - g \end{cases}$$

$\begin{matrix}\sin\theta \\ \cos\theta\end{matrix} \Rightarrow r(\sin\theta^2 + \cos\theta^2)\,\theta' = v\cos\theta - u\sin\theta$

$$\theta' = \frac{1}{r}(v\cos\theta - u\sin\theta)$$

(A) $u' = \frac{d}{dt}(-r\sin\theta\,\theta') = -r\left(\cos\theta\,(\theta')^2 + \sin\theta\,\theta''\right) = \mu\, r\cos\theta$ $\quad(-\sin\theta)$

(B) $v' = \frac{d}{dt}(r\cos\theta\,\theta') = r\left(-\sin\theta\,(\theta')^2 + \cos\theta\,\theta''\right) = \mu\, r\sin\theta - g$ $\quad(\cos\theta)$

$$-(A)\sin\theta + (B)\cos\theta = \underline{r\,\theta'' = -g\cos\theta}$$

SIMPLE CASES

COORDINATE PARTITION FOR INDEX-1 DAE IN SEMI-EXPLICIT FORM

$x = (x_1\ x_2\ \ x_m)$
$y = (y_1\ y_2\ \ y_m)$

$$\begin{cases} x' = f(x, y) \\ 0 = g(x, y) \end{cases}$$ where $\frac{\partial g}{\partial y}(x,y)$ is square and non singular

Can use implicit function to write $y(x)$

$g(x,y) = 0 \Rightarrow$ find $y(x)$ such that $g(x, y(x)) = 0$

$\Rightarrow$ FORMALLY

$x' = f(x, y(x))$

$y(x) =$ the solution of $g(x, y) = 0$

COORDINATE PARTITION FOR INDEX-2 DAE IN SEMI EXPLICIT FORM

$$\begin{cases} x' = f(x,y) \\ 0 = g(x) \end{cases}$$

$\frac{\partial g}{\partial x}$ must be full rank

$\Rightarrow$ we can partion x in 2 set x_1 x_2 such that

$$g(x) = g(x_1 x_2) = 0$$

$$x_2(x_1) \Rightarrow g(x_1, x_2(x_1)) = 0$$

$$x' = f(x,y) \Rightarrow \begin{cases} x'_1 = f_1(x_1 x_2, y) \\ x'_2 = f_2(x_1 x_2 y) \end{cases}$$

$$x'_2 = \frac{d}{dt} x_2(x_1(t)) = \frac{\partial x_2}{\partial x_1} x'_1(t)$$

$$= \frac{\partial x_2}{\partial x_1} f_1(x_1 x_2, y)$$

$$\frac{\partial x_2}{\partial x_1} f_1(x_1, x_2(x_1), y) = f_2(x_1, x_2(x_1), y)$$

$\Rightarrow$ ALGEBAIC EQUATION FOR Y

Example with index-2 DAE

$$\begin{cases} x' = -y \\ 0 = x - \sin(t) \end{cases}$$

$x' - \cos(t) = 0 \quad -y - \cos(t) = 0$

$-y' + \sin(t) = 0 \Rightarrow$ INDEX-2

independent $\Rightarrow$ empty set

dependent $x \Rightarrow x(t) = \sin t$

$x' = \cos t \Rightarrow y = -\cos t$

$\Rightarrow$ PATOLOGICAL EXAMPLE

ALGEBRAIC EQUATION

MASKED TO AN INDEX-2 DAE