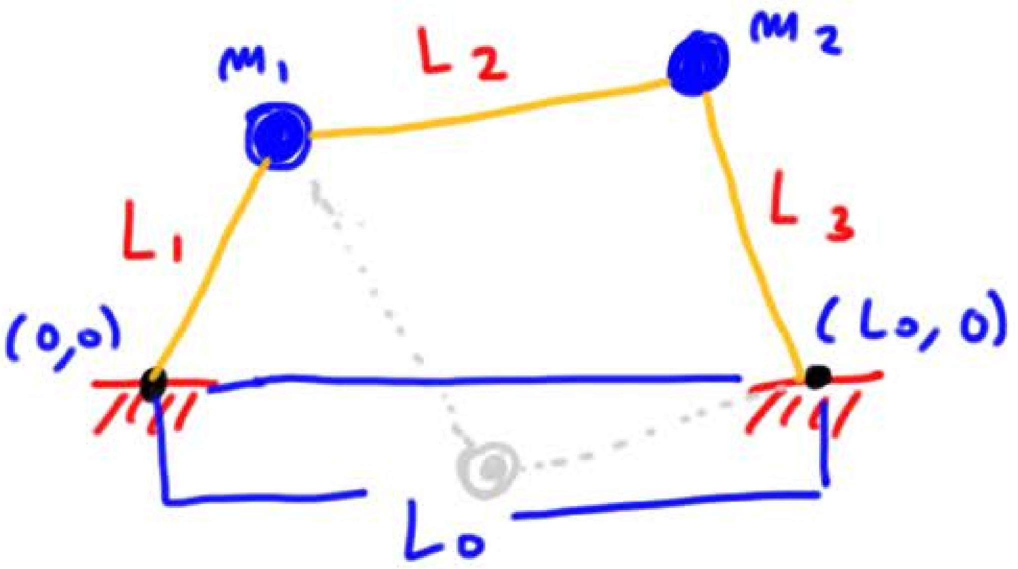
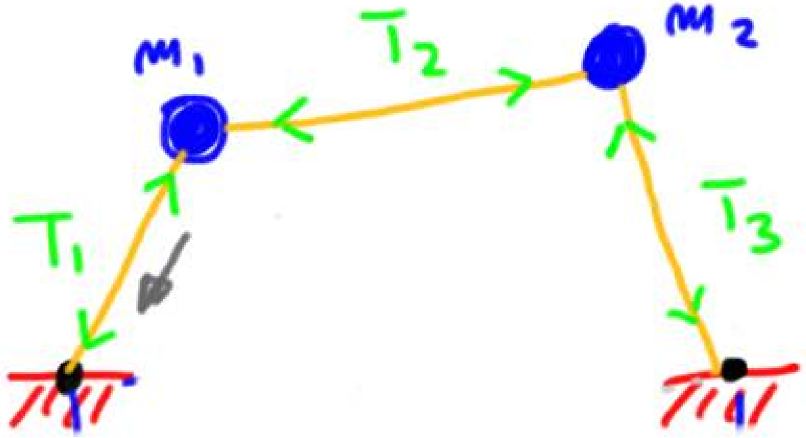
QUADRILATERAL EXAMPLE



$$\left.\begin{aligned} m_1 x_1'' &= f_{x_1} \\ m_1 y_1'' &= f_{y_1} \\ m_2 x_2'' &= f_{x_2} \\ m_2 y_2'' &= f_{y_2} \end{aligned}\right\} \text{differential part}$$

$$\left.\begin{aligned} x_1^2 + y_1^2 &= L_1^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 &= L_2^2 \\ (x_2 - L_0)^2 + y_2^2 &= L_3^2 \end{aligned}\right\} \text{Algebraic constraints}$$



$$\begin{pmatrix} f_{x_1} \\ f_{y_1} \end{pmatrix} = \frac{T_1}{L_1} \begin{pmatrix} -x_1 \\ -y_1 \end{pmatrix} + \frac{T_2}{L_2} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$\begin{pmatrix} f_{x_2} \\ f_{y_2} \end{pmatrix} = -\frac{T_2}{L_2} \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \frac{T_3}{L_3} \begin{pmatrix} x_2 - L_0 \\ y_2 \end{pmatrix}$$

THE DAE

$$
\begin{cases}
m_1 x_1'' = -\frac{T_1}{L_1} x_1 + \frac{T_2}{L_2}(x_2 - x_1) \\
m_1 y_1'' = -\frac{T_1}{L_1} y_1 + \frac{T_2}{L_2}(y_2 - y_1) \\
m_2 x_2'' = \frac{T_2}{L_2}(x_1 - x_2) + \frac{T_3}{L_3}(x_2 - L_0) \\
m_2 y_2'' = \frac{T_2}{L_2}(y_1 - y_2) + \frac{T_3}{L_3} y_2 \\
x_1^2 + y_1^2 = L_1^2 \\
(x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2 \\
(x_2 - L_0)^2 + y_2^2 = L_3^2
\end{cases}
$$

$$x_1' = u_1$$

$$m_1 u_1' = m_1 x_1'' = \text{——}$$

$$y_1' = v_1$$

$$m_1 v_1' = m_1 y_1'' = \text{——}$$

$$x_2' = u_2$$

$$m_2 u_2' = m_2 x_2'' = \text{——}$$

$$y_2' = v_2$$

$$m_2 v_2' = m_2 y_2'' = \text{——}$$

TRANSPORT TO FIRST ORDER DAE ADDING new FUNCTIONS $u_1 v_1$, $u_2 v_2$

THE DAE

$$
\begin{cases}
m_1 x_1'' = -\dfrac{T_1}{L_1} x_1 + \dfrac{T_2}{L_2}(x_2 - x_1) \\
m_1 y_1'' = -\dfrac{T_1}{L_1} y_1 + \dfrac{T_2}{L_2}(y_2 - y_1) \\
m_2 x_2'' = \dfrac{T_2}{L_2}(x_1 - x_2) + \dfrac{T_3}{L_3}(x_2 - L_0) \\
m_2 y_2'' = \dfrac{T_2}{L_2}(y_1 - y_2) + \dfrac{T_3}{L_3} y_2 \\
\boxed{\begin{array}{l} x_1^2 + y_1^2 = L_1^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2 \\ (x_2 - L_0)^2 + y_2^2 = L_3^2 \end{array}}
\end{cases}
$$

COMPUTE THE INDEX

Apply 1° derivation

$$2x_1 x_1' + 2y_1 y_1' = 0$$

$$2(x_1 - x_2)(x_1' - x_2') + 2(y_1 - y_2)(y_1' - y_2') = 0$$

$$2(x_2 - L_0)x_2' + 2y_2 y_2' = 0$$

$$\Downarrow$$

$$
\boxed{\begin{array}{l}
2x_1 u_1 + 2y_1 v_1 = 0 \\
2(x_1 - x_2)(u_1 - u_2) + 2(y_1 - y_2)(v_1 - v_2) = 0 \\
2(x_2 - L_0)u_2 + 2y_2 v_2 = 0
\end{array}}
$$

AFTER 1 derivation

$\leadsto$ ODE FOR T_1 T_2 T_3

$$2x_1 u_1 + 2y_1 v_1 = 0$$

$$2(x_1 - x_2)(u_1 - u_2) + 2(y_1 - y_2)(v_1 - v_2) = 0$$

$$2(x_2 - L_0)u_2 + 2y_2 v_2 = 0$$

$\Downarrow$ 1 more derivative

$$2x_1' u_1 + 2x_1 u_1' + 2y_1' v_1 + 2y_1 v_1' = 0$$

$$2(x_1' - x_2')(u_1 - u_2) + 2(x_1 - x_2)(u_1' - u_2') + 2(y_1' - y_2')(v_1 - v_2)$$
$$+ 2(y_1 - y_2)(v_1' - v_2') = 0$$

$$2x_2' u_2 + 2(x_2 - L_0)u_2' + 2y_2' v_2 + 2y_2 v_2' = 0$$

$\Downarrow$

$$2u_1^2 + 2x_1 u_1' + 2v_1^2 + 2y_1 v_1' = 0$$

$$2(u_1 - u_2)^2 + 2(x_1 - x_2)(u_1' - u_2') + 2(v_1 - v_2)^2 + 2(y_1 - y_2)(v_1' - v_2') = 0$$

$$2u_2^2 + 2(x_2 - L_0)u_2' + 2v_2^2 + 2y_2 v_2' = 0$$

$$2u_1^2 + 2x_1 u_1' + 2v_1^2 + 2y_1 v_1' = 0$$

$$2(u_1-u_2)^2 + 2(x_1-x_2)(u_1'-u_2') + 2(v_1-v_2)^2 + 2(y_1-y_2)(v_1'-v_2') = 0$$

$$2u_2^2 + 2(x_2-L_0)u_2' + 2v_2^2 + 2y_2 v_2' = 0$$

$$u_1' = \frac{1}{m_1}\left(\frac{T_2}{L_2}(x_2-x_1) - \frac{T_1}{L_1}x_1\right)$$

$$v_1' = \frac{1}{m_1}\left(\frac{T_2}{L_2}(y_2-y_1) - \frac{T_1}{L_1}y_1\right)$$

$$u_2' = \frac{1}{m_2}\left(\frac{T_2}{L_2}(x_1-x_2) + \frac{T_3}{L_3}(x_2-L_0)\right)$$

$$v_2' = \frac{1}{m_2}\left(\frac{T_2}{L_2}(y_1-y_2) + \frac{T_3}{L_3}y_2\right)$$

AFTER SUBSTITUTION WE OBTAIN AN ALGEBRAIC EQUATION which contain T_1 T_2 T_3

AFTER ANOTHER DERIVATION WE CAN SOLVE

$T_1' = \text{—}$ $T_2' = \text{—}$ $T_3' = \text{—}$

CONCLUSION ?

$\Rightarrow$ IS AN INDEX 3 DAE

THE DAE WRITTEN IN VECTORIAL FORM

$$\begin{cases} z' = F(z, w) \\ 0 = H(z, w) \end{cases} \qquad \begin{aligned} z^T &= (x_1\ u_1\ x_2\ u_2\ y_1\ v_1\ y_2\ v_2) \\ w^T &= (T_1\ T_2\ T_3) \end{aligned}$$

$$z_1' = F_1(z, w) \Rightarrow \quad x_1' = u_1$$

$$z_2' = F_2(z, w) \Rightarrow \quad u_1' = \frac{1}{m_1}\left(\frac{T_2}{L_2}(x_2 - x_1) - \frac{T_1}{L_1} x_1\right)$$

$$\Downarrow$$

$$F_2(z, w) = \frac{1}{m_1}\left(\frac{w_2}{L_2}(z_3 - z_1) - \frac{w_1}{L_1} z_1\right)$$

In our example $H(z, w) \equiv H(z)$ constraints do not contain explicit dependence on $T_1\ T_2\ T_3$

$$\begin{cases} z' = F(z,w) \\ 0 = H(z,w) \end{cases} \qquad z^T = (x_1\ u_1\ x_2\ u_2\ y_1\ v_1\ y_2\ v_2) \qquad w^T = (T_1\ T_2\ T_3)$$

But $G(z,w) \equiv G(x_1\ x_2\ y_1\ y_2)$

$$\{x_1\ x_2\ y_1\ y_2\} \subset \{x_1\ u_1\ x_2\ u_2\ y_1\ v_1\ y_2\ v_2\}$$

So

$$Y = (u_1\ u_2\ v_1\ v_2) \quad Z = (x_1\ x_2\ y_1\ y_2) \quad W = (T_1\ T_2\ T_3)$$

$$\begin{cases} Y' = F(Y, Z\ W) \\ Z' = Y \\ \boxed{0 = H(Z)} \end{cases} \qquad H(Z) = \begin{cases} H_1(x_1 x_2 y_1 y_2) = 0 \\ H_2(x_1 x_2 y_1 y_2) = 0 \\ H_3(x_1 x_2 y_1 y_2) = 0 \end{cases}$$

$$H_1(x_1\ x_1\ y_1\ y_2) = x_1^2 + y_1^2 - L_1^2$$

3 EQ 4 VARS $\Rightarrow$ The solutions are 1 parameter family

$$H(z) = \begin{cases} H_1(x_1 x_2 y_1 y_2) = 0 \\ H_2(x_1 x_2 y_1 y_2) = 0 \\ H_3(x_1 x_2 y_1 y_2) = 0 \end{cases}$$

Four choice for coordinate partition

Independent set I	dependent set D
x_1	$\{x_2 y_1 y_2\}$
y_1	$\{x_1 x_2 y_2\}$
x_2	$\{x_1 y_1 y_2\}$
y_2	$\{x_1 x_2 y_1\}$

IF x_1 is FIXED $x_2 y_1 y_2$ are "determined"

$\Rightarrow x_2(x_1)\ y_1(x_1)\ y_2(x_1)$

IF y_1 IS FIXED $x_1\ x_2\ y_2$ are "determined"

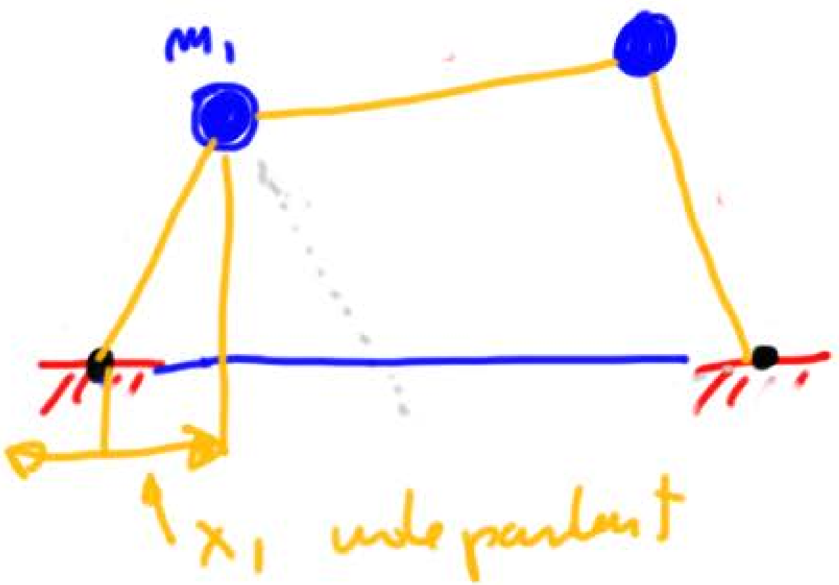
$\Rightarrow x_1(y_1)\ x_2(y_1)\ y_2(y_1)$

IF x_2 IS FIXED $x_1 y_1 y_2$ are "determined"

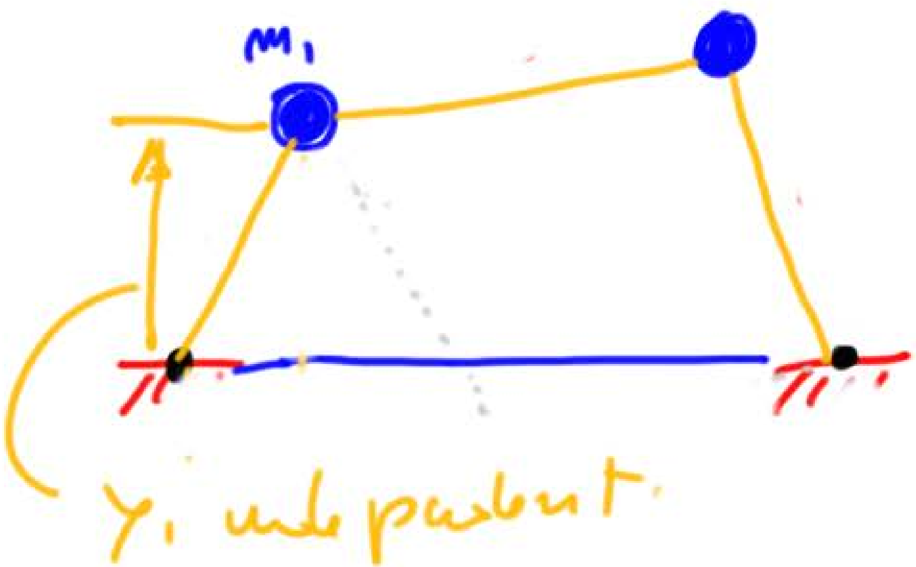
$\Rightarrow x_1(x_2)\ y_1(x_2)\ y_2(x_2)$

IF y_2 IS FIXED $x_1\ x_2\ y_1$ "determines"

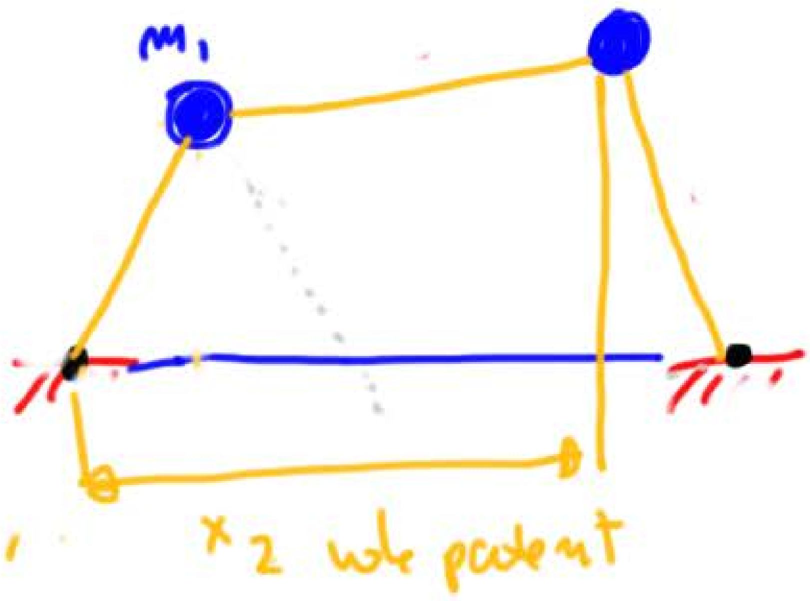
$\Rightarrow x_1(y_2)\ x_2(y_2)\ y_1(y_2)$



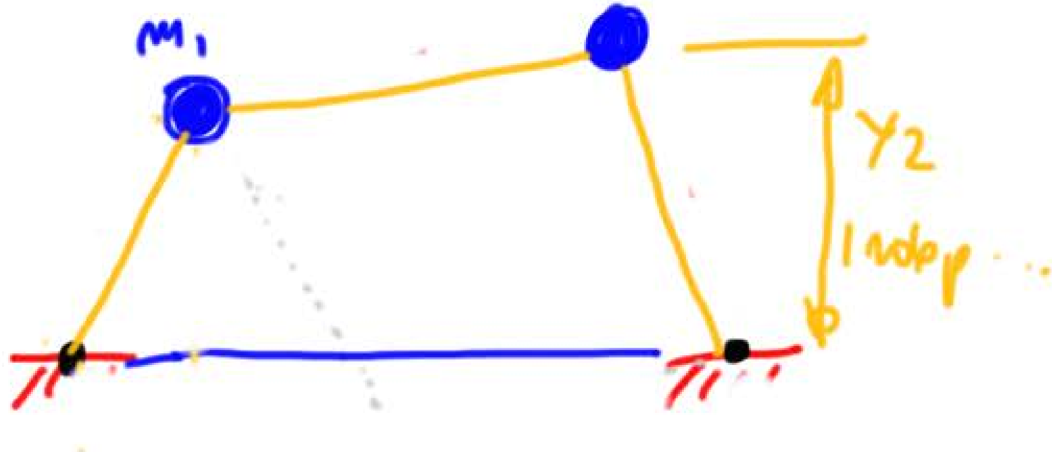
m1
x1 independent



m1
Y1 independent.



m1
x2 independent



m1
Y2
Indep...

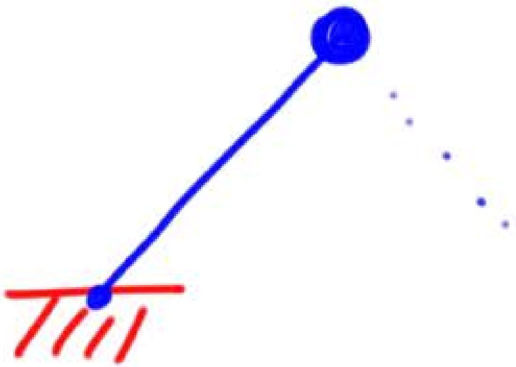
Using x_1 as independent coordinate

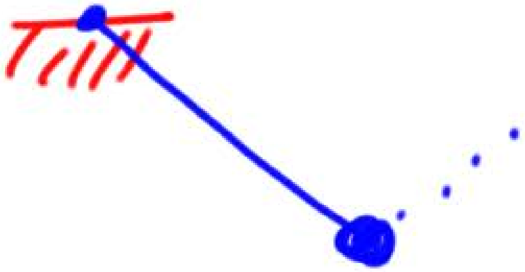
$$x_1^2 + y_1^2 = L_1^2$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2$$

$$(x_2 - L_0)^2 + y_2^2 = L_3^2$$

$$\Rightarrow y_1(x_1) = +\sqrt{L_1^2 - x_1^2} \qquad y_1(x_1) = -\sqrt{L_1^2 - x_1^2}$$





$$(x_2 - L_0)^2 + y_2^2 = L_3^2$$

$$y_2(x_2) = \pm\sqrt{L_3^2 - (x_2 - L_0)^2} \qquad x_2(y_2) = L_0 \pm \sqrt{L_3^2 - y_2^2}$$

$$x_1^2 + y_1^2 = L_1^2$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2$$

$$(x_2 - L_0)^2 + y_2^2 = L_3^2$$

$$y_1(x_1) = \pm\sqrt{L_1^2 - x_1^2}$$

$$y_1(x_1) = \pm\sqrt{L_1^2 - x_1^2} \qquad x_1(y_1) = \pm\sqrt{L_1^2 - y_1^2}$$

$$y_2(x_2) = \pm\sqrt{L_3^2 - (x_2 - L_0)^2} \qquad x_2(y_2) = L_0 \pm\sqrt{L_3^2 - y_2^2}$$

$$(x_1 - x_2)^2 + (y_1(x_1) - \underline{y}_2(x_1))^2 = L_2^2$$

$$x_1(x_1) = x_1 \pm\sqrt{L_2^2 - (y_1(x_1) - y_2(x_1))^2}$$

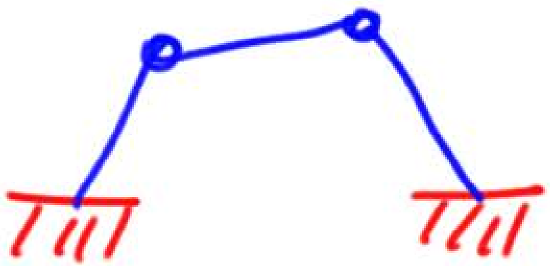
<u>As an example</u>

X_1	X_2	Y_1	Y_1
+	+	+	+
−	+	+	+
+	−	+	+
−	−	+	+
+	+	−	+
−	+	−	+
+	−	−	+
−	−	−	+
+	+	+	−
−	+	+	−
+	−	+	−
−	−	+	−
+	+	−	−
−	+	−	−
+	−	−	−
−	−	−	−

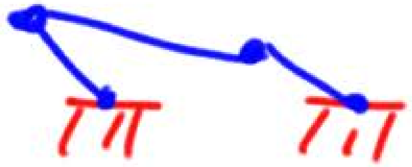
16 configuration

× 4 choice of independent variable (divided by 2)

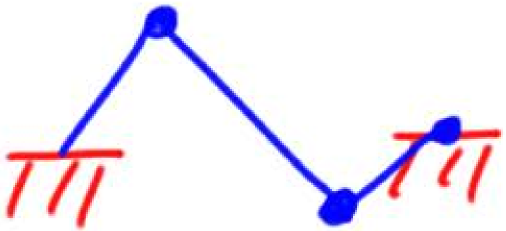
$\frac{16 \times 4}{2} = 32$ maps



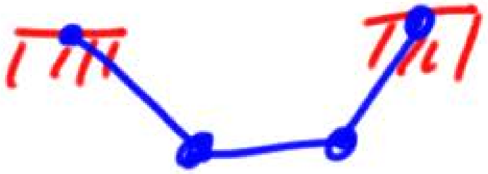
x_1	y_1	x_2	y_2
+	+	+	+



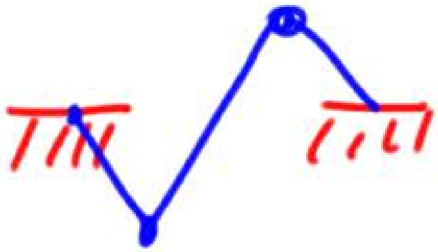
x_1	y_1	x_2	y_2
−	+	+	+



x_1	y_1	x_2	y_2
+	+	+	−



x_1	y_1	x_2	y_2
+	−	+	−



x_1	y_1	x_2	y_2
+	−	+	+

may be $x_1 < 0$
impossible

or $x_1 < 0$ may
be possible
depend on $L_0 L_1 L_2 L_3$

DO FULL COMPUTATION FOR CONFIGURATION

$\pm + + +$

x_1 independent

$$\left.\begin{aligned} x_1^2 + y_1^2 &= L_1^2 \\ (x_1 - x_2)^2 + (y_1 - y_2)^2 &= L_2^2 \\ (x_2 - L_0)^2 + y_2^2 &= L_3^2 \end{aligned}\right\} \Rightarrow \begin{aligned} y_1(x_1) &= \sqrt{L_1^2 - x_1^2} \\ y_2(x_2) &= \sqrt{L_3^2 - (x_2 - L_0)^2} \end{aligned}$$

$$\Downarrow$$

$$(x_1 - x_2)^2 + (y_1(x_1) - y_2(x_2))^2 = L_2^2$$

$$\Downarrow$$

Too complex algebraic expression

PRACTICAL COMPUTATION

$$x_1^2 + y_1^2 = L_1^2$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2$$

$$(x_2 - L_0)^2 + y_2^2 = L_3^2$$

IF WE ARE SEARCHING $y_1(x_1)$ $x_2(x_1)$ $y_2(x_1)$ SIMBOLICALLY

FORGET IT!

COMPUTE $y_1(x_1)$ $x_2(x_1)$ $y_2(x_1)$ NUMERICALLY

SOLVING

$$\begin{cases} g_1(y_1\ x_2\ y_2) = x_1^2 + y_1^2 - L_1^2 \\ g_2(y_1\ x_2\ y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2 - L_2^2 \\ g_2(y_1\ x_2\ y_2) = (x_2 - L_0)^2 + y_2^2 - L_3^2 \end{cases}$$

THE SOLUTION IS $y_1(x_1)$ $y_2(x_1)$ $x_2(x_1)$

IF WE ARE ABLE TO SOLVE THE ALGEBRAIC SISTEM

$$x_1^2 + y_1^2 = L_1^2$$

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L_2^2$$

$$(x_2 - L_0)^2 + y_2^2 = L_3^2$$

WE CAN WRITE FORMALLY

x_1	$y_1(x_1)$	$x_2(x_1)$	$y_2(x_1)$
y_1	$x_1(y_1)$	$x_2(y_1)$	$y_2(y_1)$
x_2	$x_1(x_2)$	$y_1(x_2)$	$y_2(x_2)$
y_2	$x_1(y_2)$	$y_1(y_2)$	$x_2(y_2)$

IMPORTANT

$x_1(t)$
$x_2(t)$
$y_1(t)$
$y_2(t)$
$\vdots$

THIS FUNCTION OF t are the solution of the DAE

when we write

$y_1(x_1)$

THIS ARE TOTALLY DIFFERENT FUNCTION with the same name but correlated

$$y_1(t) = y_1(x_1(t))$$

CONSEQUENCES

$$m_1 v_1'(t) = -\frac{T_1(t)}{L_1} y_1(t) + \frac{T_2(t)}{L_2}\left(y_2(t) - y_1(t)\right)$$

BUT USING ALGEBRAIC PART WE CAN COMPUTE $y_1(x_1)$ $x_2(x_1)$ $y_2(x_1)$ so that

$$y_1(t) = Y_1(x_1(t))$$

$$y_2(t) = Y_2(x_1(t))$$

$$v_1(t) = \frac{d}{dt} y_1(t) = \frac{d}{dt} Y_1(x_1(t))$$

$$= \frac{\partial Y_1}{\partial x}(x_1(t))\, x_1'(t)$$

m

$$v_1'(t) = \frac{d}{dt}\left[\frac{\partial Y_1}{\partial x}(x_1(t))\, x_1'(t)\right]$$