SEMI-EXPLICIT DAE

$$\begin{cases} x' = f(x,y,t) \\ 0 = g(x,y,t) \end{cases}$$

Add the differential equation

$t'(s) = 1$ and

interpret the DAE as a function of s

$$\begin{cases} x'(s) = f(x(s), y(s), t(s)) \\ t'(s) = 1 \\ 0 = g(x(s), y(s), t(s)) \end{cases}$$

ANY ODE OR DAE CAN BE TRANSFORMED IN AN AUTONOMOUS ODE OR DAE

$$z = \begin{pmatrix} x \\ t \end{pmatrix} \quad F(z,y) = \begin{pmatrix} f(x,y,t) \\ 1 \end{pmatrix}$$

$$G(z,y) = g(x,y,t)$$

$$\begin{cases} z' = F(z,y) \\ 0 = G(z,y) \end{cases}$$

SEMI-EXPLICIT DAE (AUTONOMOUS)

$$\begin{cases} x' = f(x,y) \\ 0 = g(x,y) \end{cases}$$

IT IS POSSIBLE TO
DO ALL THE COMPUTATION
IN THE AUTONOMOUS CASE
FOR EXAMPLE SETTING
$x_1' = 1$ (which is the
ONE FOR THE TIME)

AND AT THE END ELIMINATE
$x_1(t) = t$

THIS SIMPLIFY EXPOSITION

SEMI-EXPLICIT DAE (CATALOGUE)

$$\begin{cases} x' = f(x,y) \\ 0 = g(x,y) \end{cases}$$

independent (x), dependent (y)

and $\dfrac{\partial g}{\partial y}(x,y)$ non singular so that by using implicit function theorem

$$y = y(x)$$

$\Rightarrow$ THE DAE IS INDEX-1 DAE THAT CAN BE REDUCED TO AN ODE

$$x' = f(x, y(x))$$

TRIVIAL EXAMPLE OF COORDINATE PARTITION TECHNIQUE

SEMI-EXPLICIT DAE (CATALOGUE)

$$\begin{cases} x' = f(x,y) \\ 0 = g(x,y) \end{cases} \quad \text{and} \quad \frac{\partial g}{\partial y}(x,y)$$

SINGULAR

$\Rightarrow y = y(x)$ NO!

A VERY SINGULAR $\frac{\partial g}{\partial y}$ is when $\frac{\partial g}{\partial y} = 0$

so we consider $g(x,y) = g(x)$

$$\begin{cases} x' = f(x,y) \\ 0 = g(x) \end{cases} \qquad \frac{\partial g}{\partial x}(x)\, x' = 0 \quad \frac{\partial g}{\partial x}(x)\, f(x,y) = 0$$

$$h(x,y) = \frac{\partial g}{\partial x}(x)\, f(x,y)$$

$$\boxed{\frac{\partial g}{\partial x}\frac{\partial f}{\partial y}}$$

condition of index-2

$\Downarrow$

$$\begin{cases} x' = f(x,y) \\ 0 = h(x,y) \end{cases} \quad \text{if } \frac{\partial h}{\partial y} \text{ is}$$

NONSINGULAR

$\Rightarrow$ IS AN INDEX-1 DAE

IS AN INDEX-2 DAE

$$\begin{cases} x' = f(x,y) \\ 0 = g(x) \end{cases}$$

$$\frac{\partial g}{\partial x}(x)\, x' = \frac{\partial g}{\partial x}(x)\, f(x,y) = h(x,y)$$

if $\frac{\partial h}{\partial y}$ SINGULAR, CANNOT BE INDEX-2 DAE

$$\frac{\partial h}{\partial y}(x,y) = \frac{\partial g}{\partial x}(x)\,\frac{\partial f}{\partial y}(x,y)$$

NOT of maximum RANK

$$g(x) = g(x_1, x_2) = g(x_2)$$

g does not depend on x_1

$$\begin{cases} x_1' = f_1(x_1, x_2, y) \\ x_2' = f_2(x_1, x_2, y) \\ 0 = g(x_1) \end{cases}$$

DAE of index more than $\underline{\underline{2}}$

SEMI-EXPLICIT INDEX-3 DAE

ARE (NORMALLY WRITTEN AS)

$$\begin{cases} x' = f(x,y,\lambda) \\ y' = g(x,y) \\ 0 = h(y) \end{cases}$$

$$\frac{\partial h}{\partial y}(y)\,\frac{\partial g}{\partial x}(x,y)\,\frac{\partial f}{\partial \lambda}(x,y,\lambda) = \text{NONSINGULAR}$$

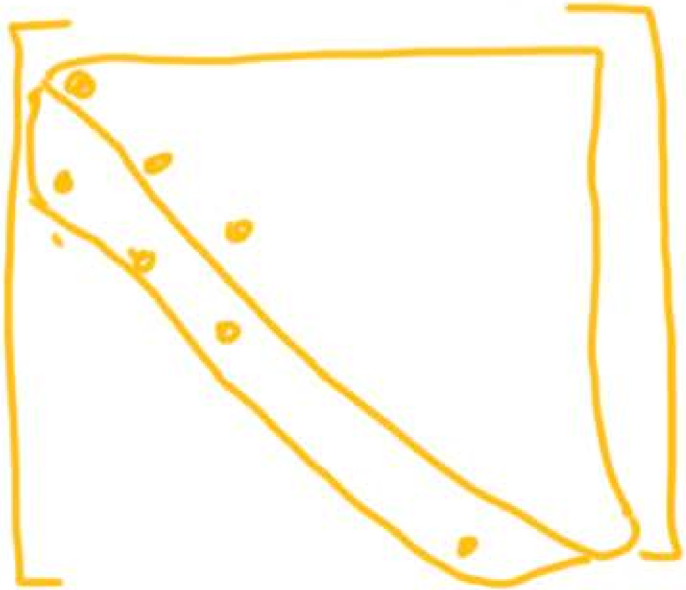
$\Rightarrow$ INDEX-1 DAE

SEMI-EXPLICIT DAE of INDEX N

$$
\begin{cases}
x_1' = f_1(x_1, x_2, \ldots, x_n) \\
x_2' = f_2(x_1, x_2, \ldots, x_{n-1}) \\
x_3' = f_3(x_2, x_3, \ldots, x_{n-1}) \\
\vdots \\
x_{n-1}' = f_{n-1}(x_{n-1}) \\
0 = f_n(x_{n-1})
\end{cases}
$$

HESSENBERG FORM

$$
\underbrace{\frac{\partial f_n}{\partial x_{n-1}} \, \frac{\partial f_{n-1}}{\partial x_{n-2}} \cdots \frac{\partial f_2}{\partial x_1} \, \frac{\partial f_1}{\partial x_n}}_{\text{NON SINGULAR}}
$$



HESSENBERG MATRIX

COORDINATE PARTITION FOR INDEX-3 DAE (SEMI EXPLICIT)

$$\begin{cases} x' = f(x, y, \lambda) \\ y' = g(x, y) \\ 0 = h(y) \end{cases}$$

$\leftarrow$ PARTITION $y = (y_1, y_2)$ so that $h(y_1, y_2) = 0$ can be solved (by implicit function theorem) $y_2 = y_2(y_1)$

$h(y_1, y_2(y_1)) = 0 \Rightarrow$ can be eliminated

$$\begin{cases} x' = f(x, y_1, y_2(y_1), \lambda) \\ y_1' = g_1(x, y_1, y_2(y_1)) \\ \frac{d}{dt} y_2(y_1) = g_2(x, y_1, y_2(y_1)) \end{cases}$$

$$\begin{cases} x' = f(x, y_1, y_2(y_1), \lambda) \\ y_1' = g_1(x, y_1, y_2(y_1)) \\ \dfrac{d}{dt} y_2(y_1) = g_2(x, y_1, y_2(y_1)) \end{cases}$$

$\Downarrow$

$$\frac{d}{dt} h(y_1, y_2(y_1)) = \frac{\partial h}{\partial y_1}(\ ) y_1' + \frac{\partial h}{\partial y_2} \frac{\partial y_2}{\partial y_1}(y_1)\, y_1' = 0$$

$$\frac{d}{dt} y_2(y_1) = \frac{\partial y_2}{\partial y_1}(y_1)\, y_1' \Rightarrow \frac{d}{dt} y_2(y_1) = -\left(\frac{\partial h}{\partial y_2}\right)^{-1} \frac{\partial h}{\partial y_1} y_1'$$

$$-\left(\frac{\partial h}{\partial y_2}\right)^{-1} \frac{\partial h}{\partial y_1} g_1 = g_2 \Rightarrow \frac{\partial h}{\partial y_1} g_1 + \frac{\partial h}{\partial y_2} g_2 = 0$$

$$\left(\frac{\partial h}{\partial y_1}, \frac{\partial h}{\partial y_2}\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{\partial h}{\partial y} g = 0 \Leftarrow \text{AN ALGEBRAIC EQUATION}$$

$$\begin{cases} x' = f(x, y_1, y_2(y_1), \lambda) \\ y_1' = g_1(x, y_1, y_2(y_1)) \\ \frac{d}{dt} y_2(y_1) = g_2(x, y_1 y_2(y_1)) \end{cases} \Rightarrow \frac{\partial h}{\partial y}(y) g(x, y) = A(x, y)$$

$$\begin{cases} x' = f(x, y_1, y_2(y_1), \lambda) \\ y_1' = g_1(x, y_1, y_2(y_1)) \\ A(x, y_1, y_2(y_1)) = 0 \end{cases} \quad x = (x_1\ x_2) \text{ such that by implicit function theorem}$$

$$A(x_1, x_2(x_1, y_1, y_2), y_1, y_2) = 0$$

$$\begin{cases} x_1' = f_1(x_1, x_2(x_1, y_1, y_2(y_1)), y_1, y_2(y_1), \lambda) \\ \frac{d}{dt} x_2(x_1) = f_2(x_1\ x_2, y_1, y_2, \lambda) \\ y_1' = g_1(x, y_1, y_2(y_1)) \end{cases}$$

$$\begin{cases} x_1' = f_1(x_1, x_2(x_1, y_1, y_2(y_1)), y_1, y_2(y_1), \lambda) \\ \dfrac{d}{dt} x_2(x_1) = f_2(x_1, x_2, y_2, y_2, \lambda) \end{cases} \qquad y_1' = g_1(x, y_1, y_2(y_1))$$

$$\frac{d}{dt} x_2(x_1) = \frac{\partial x_2}{\partial x_1} x_1' = \frac{\partial x_2}{\partial x_1} f_1(x_1\, x_2\, y_1\, y_2\, \lambda)$$

$$A(x_1\, x_2\, y_1\, y_2) = 0 \qquad x_2 \equiv x_2(x_1, y_1, y_2)$$

$$A(x_1(t), x_2(x_1(t), y_1(t), y_2(y_1(t)), y_2(y_1(t))) = 0$$

$$\frac{\partial A}{\partial x_1} x_1' + \frac{\partial A}{\partial x_2}\left(\frac{\partial x_2}{\partial x_1} x_1' + \frac{\partial x_2}{\partial y_1} y_1' + \frac{\partial x_2}{\partial y_2}\frac{\partial y_2}{\partial y_1} y_1'\right)$$

$$+ \frac{\partial A}{\partial y_2}\frac{\partial y_2}{\partial y_1} y_1' = 0$$

Solve this Equation

$$\begin{cases} x_1' = f_1(x_1, x_2(x_1, y_1, y_2(y_1)), y_1, y_2(y_1), \lambda) \\ \frac{d}{dt} x_2(x_1) = f_2(x_1, x_2, y_2, y_2, \lambda) \end{cases}$$

$$y_1' = g_1(x, y_1, y_2(y_1))$$

$$\frac{d}{dt} x_2(x_1) = \frac{\partial x_2}{\partial x_1} x_1' =$$

$$A(x_1(t), x_2(x_1(t), y_1(t), y_2(y_1(t)), y_2(y_1(t))) = 0$$

$$\frac{\partial A}{\partial x_1} x_1' + \frac{\partial A}{\partial x_2}\left(\frac{\partial x_2}{\partial x_1} x_1' + \frac{\partial x_2}{\partial y_1} y_1' + \frac{\partial x_2}{\partial y_2}\frac{\partial y_2}{\partial y_1} y_1'\right) + \frac{\partial A}{\partial y_2}\frac{\partial y_2}{\partial y_1} y_1' = 0$$

IS ALGEBRAIC FROM PREVIOUS SLIDES

$$\frac{d}{dt} x_2(x_1) = B(x_1 x_2 y, y_2, \lambda)$$

(x can be solved)

$$\begin{cases} x_1' = f_1(x_1, x_2(x_1, y_1, y_2(y_1)), y_1, y_2(y_1), \lambda) \\ \frac{d}{dt} x_2(x_1) = f_2(x_1, x_2, y_2, y_2, \lambda) \end{cases}$$

$$y_1' = g_1(x, y_1, y_2(y_1))$$

$$C(x_1 x_2 y_1 y_2 \lambda) = 0$$

$$\frac{d}{dt} x_2(x_1) = B(x_1 x_2 y_1 y_2, \lambda)$$

$$\begin{cases} x_1' = f_1(x_1 x_2 y_1 y_2 \lambda) \leftarrow \underline{\underline{\text{ODE}}} \\ y_1' = g_1(x_1 x_2 y_1 y_2) \\ 0 = C(x_1 x_2 y_1 y_2 \lambda) \Rightarrow \lambda = \lambda(x_1, x_2 y_1, y_2) \end{cases}$$